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Classical Electrodynamics

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References

This Course Reader is based on and compiled from:

ŠTOLL, Ivan, Jiří TOLAR a Igor JEX. Klasická teoretická fyzika. Praha: Univerzita Karlova, nakladatelství Karolinum, 2017. ISBN 978-80-246-3545-3.

JACKSON, John D. Classical Electrodynamics. 3rd edition, John Wiley & Sons Inc., New York., 1999. ISBN: 978-0-471-30932-1

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Chapter 1

Electromagnetic field

According to today's knowledge, atoms of all substances are composed of electrically charged particles, i.e. electrons and atomic nuclei. Substances may seem electrically neutral, if the influences of opposite charges interfere with each other. We explain the electric current as motion of electrically charged particles.

The contemporary physics is based on these ideas and aims to determine the structure of substances and deduce the laws of physical, or chemical phenomena at the level of atoms and molecules from the laws of the motion of electrically charged particles. The first step in this direction is to clarify the laws of the mutual action (interaction) of electric charges through an electromagnetic field.

In this paragraph, we will formulate the basic laws of electrodynamics, i.e. Maxwell and Lorentz-Maxwell equations. These equations play the same role in electrodynamics as Newton's equations in classical mechanics. They were found by generalization of empirically established regularities. Their general validity is, however, based on the fact that all of the consequences of these laws have been experimentally proved and even found a great number of applications in technical practice.

The origin of electrodynamics dates back to the nineteenth century (M. Faraday 1831, J. C. Maxwell 1865), and its final formulation was completed only at the end of the nineteenth century (H. A. Lorentz 1892), i.e. considerably later than that of classical mechanics. Therefore, there were efforts to reduce its laws to the laws of mechanics, which seemed to be simpler. However, later development demonstrated that the electrodynamics is independent of classical mechanics, and both these disciplines constitute the two fundamental pillars for the current understanding of the structure and motion of matter.

1.1 Maxwell's equations

The fundamental laws of a non-stationary electromagnetic field induced by moving charged particles *in vacuum* can be summarized in two series of the fundamental Lorentz-Maxwell equations:

$$\text{I. série:} \qquad \mathrm{div}\boldsymbol{E} = \frac{\rho}{\varepsilon_0}, \quad \mathrm{rot}\boldsymbol{B} - \varepsilon_0\mu_0\frac{\partial \boldsymbol{E}}{\partial t} = \mu_0 \boldsymbol{j}$$

$$\text{II. série:} \qquad \mathrm{rot}\boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0, \qquad \mathrm{div}\boldsymbol{B} = 0 \tag{1.1}$$

(Hendrik Antoon Lorentz in 1892).

These partial differential equations of the first order determine the spatial distribution and temporal changes of the electromagnetic field induced in vacuum by particles at *given density of electric charge* $\rho(\boldsymbol{r}, t)$ and *current density* $\boldsymbol{j}(\boldsymbol{r}, t)$. Here $\boldsymbol{j} = \rho\boldsymbol{v}$, where $\boldsymbol{v}(\boldsymbol{r}, t)$ is the distribution of velocity of charges' motion. The electromagnetic field is determined by the distribution of vector fields of *electric field intensity* $\boldsymbol{E}(\boldsymbol{r}, t)$ and *magnetic induction* $\boldsymbol{B}(\boldsymbol{r}, t)$, which are the solution of the set of equations (1.1). Vacuum permittivity ε_0 and permeability μ_0 are two constants that have been introduced in the International System of Units (SI) for physical quantities. They have approximate values

$$\varepsilon_0 \doteq 8{,}854 \cdot 10^{-12}\ \mathrm{Fm}^{-1}, \qquad \mu_0 \doteq 1{,}257 \cdot 10^{-6}\ \mathrm{Hm}^{-1}\ ,$$

(where F denotes the farad unit, H denotes henry), and fulfill the important Weber relation (Wilhelm Eduard Weber, 1804 1891)

$$\varepsilon_0\ \mu_0 = \frac{1}{c^2}, \tag{1.2}$$

where $c \doteq 2,998 \cdot 10^8$ m s^{-1} is the speed of light in vacuum. Unlike the artificially established constants ε_0 and μ_0, c is a natural constant whose value has to be determined experimentally. We can, of course, turn from the SI system to another system (e.g. to the so-called 'absolute' system), where constants ε_0 and μ_0 do not appear. We cannot, however, dispense with the constant c. The quantities $\boldsymbol{E}$, $\boldsymbol{B}$, c in the Lorentz-Maxwell equations (1.1) indicate the connection between electricity, magnetism and light.[1]

The equations (1.1) determine the electromagnetic field if the distribution of ρ, $\boldsymbol{j}$ is given. The other side of the charged particle interaction is the force of the electromagnetic field $\boldsymbol{E}$, $\boldsymbol{B}$ acting on the charges. It is determined by a known formula for the **Lorentz force**

$$\boldsymbol{F} = q\ (\boldsymbol{E}\ +\ \boldsymbol{v}\ \times \boldsymbol{B}), \tag{1.3}$$

which acts on a particle with a charge q moving at a speed $\boldsymbol{v}$. This force is used in many technical devices including electric motors, accelerators, mass spectrometers, particle separators, etc. The formula (1.3) determines, together with the equation of motion of (relativistic) mechanics, the motion of a charged particle with the mass at rest m_0 in the given field $\boldsymbol{E}$, $\boldsymbol{B}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m_0\ \boldsymbol{v}}{\sqrt{1-\frac{v^2}{c^2}}}\right)\ =\ \boldsymbol{F} = q\ (\boldsymbol{E}\ +\ \boldsymbol{v}\ \times \boldsymbol{B}). \tag{1.4}$$

In the first half of the nineteenth century, it was believed that the electromagnetic force can be reduced to mechanics and Newton's equations (the 'mechanical picture of the world'). The discovery of electron in 1897 led to the formation of an opinion that the substance is actually a set of charged particles moving in a vacuum, and Lorentz-Maxwell equations are sufficient enough to describe their properties (the 'electromagnetic picture of the world', the Lorentz electron theory). As we know today, both of these approaches are one-sided, and the reality is more complex. Modern microscopic physics is primarily based on the connection and combination of quantum theory with special relativity.

If we want to describe the motion of a system of charged particles, we have to realize that the moving particles induce the electromagnetic field according to the expression 1.1, but at the same time they move in this field according to the relation (1.4). Thus, in order to provide the complete description of phenomenon, we need to solve a nonlinear problem, the Lorentz-Maxwell equations along with the motion equations of relativistic mechanics. A satisfactory general solution is yet to be found for this problem.

In applications, we most often encounter the task of determining the electromagnetic field in the presence of macroscopic bodies at rest. In such circumstances, we are usually not interested in the local progress of the microscopic field in atomic dimensions (of the order of 10^{-10} m), but only in macroscopic mean values in space and time. Our measuring instruments also show us these mean values. If we say that some macroscopic body is not electrically charged, it does not mean that there are no microscopic electric charges in its volume. On the contrary, very strong electromagnetic fields, which are rapidly variable in time, are induced inside atoms and atomic nuclei. However, in the process of space-time averaging, the effects of these charges interfere.

Using the mathematical procedure of averaging the Maxwell-Lorentz equations (see e.g. [22]) for a local field over the areas small enough in terms of macroscopic accuracy (10^{-4} to 10^{-2} m), but containing a sufficient number of molecules, we derive Maxwell's equations

$$\text{I. série} \qquad \mathrm{div}\boldsymbol{D} = \rho, \quad \mathrm{rot}\boldsymbol{H} - \frac{\partial \boldsymbol{D}}{\partial t} = \boldsymbol{j}$$

$$\text{II. série} \qquad \mathrm{rot}\boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0, \qquad \mathrm{div}\boldsymbol{B} = 0 \tag{1.5}$$

(James Clerk Maxwell in 1865). [2]

Note that in a *stationary* case (when $\partial/\partial t = 0$), the system of equations (1.5) falls into two independent systems, one of which describes the electric field and another one the magnetic field. On the contrary, in a *non-stationary case*, these fields are no longer independent; we are therefore speaking of an electromagnetic field.

Maxwell's equations (1.5) determine the electromagnetic field in *material medium* with the specified distribution of sources, i.e. the *free* charges density $\rho(\boldsymbol{r}, t)$ and the *free* current density $\boldsymbol{j}(\boldsymbol{r}, t)$. Besides the primary quantities $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{B}(\boldsymbol{r}, t)$ that determine the electromagnetic field, secondary quantities, i.e. *electrical induction* $\boldsymbol{D}(\boldsymbol{r}, t)$ and *intensity* of *magnetic field*

[1] In the SI system the speed of light in vacuum was determined by the exact value c = 299 792 458 m s^{-1}, then numerically $\mu_0 = 4\pi \cdot 10^{-7}$ and hence $\varepsilon_0 = 1/\mu_0 c^2$. If any future measurements would further refine the magnitude of speed of light in vacuum, its defined numeric value would remain preserved, and we would rather change the length of one meter!

[2] Ludvig Valentin Lorenz (1829-1891), a Danish physicist, independent of J. C. Maxwell, but two years later (1867), published the same set of equations for the electromagnetic field. His fundamental contribution to the theory of electromagnetism had been largely forgotten for a long time.

H *(**r**,* t) appear there. They also include the influence of inaccessible bound charges and currents induced in the substance.

They are described by additional *defining relations*

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P} \ , \qquad \boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_0} - \boldsymbol{M} \ . \tag{1.6}$$

Here ***P***(***r***, *t*) is called the *polarization vector* (it indicates the total electric dipole moment of the volume unit) and ***M***(***r***, *t*) is called the *vector of magnetization* (indicates the total magnetic dipole moment of the volume unit). In general, these quantities depend both on the internal construction of the substance and on the values of the fields ***E***, ***B***. For particular substances, it is necessary to determine these dependencies, the so-called *material relations,* experimentally.

If the mean fields ***E***, ***B*** are weak (relative to local fields), for isotropic substances the following linear relations are valid with good accuracy

$$\boldsymbol{P} = \varepsilon_0 \, \chi \, \boldsymbol{E}, \qquad \boldsymbol{M} = \varkappa \, \boldsymbol{H} \ , \tag{1.7}$$

where χ and $\varkappa$ indicate *electrical and magnetic susceptibility.* By substituting into the expression (1.6), we receive relations

$$\boldsymbol{D} = \varepsilon \boldsymbol{E} \ , \qquad \boldsymbol{H} = \frac{\boldsymbol{B}}{\mu} \ , \tag{1.8}$$

where ε is *the permittivity* and μ *is the permeability* of the substance:

$$\varepsilon = \varepsilon_0 \varepsilon_r \ , \qquad \varepsilon_r = 1 + \chi \ , \qquad\qquad \mu = \mu_0 \mu_r \ , \qquad \mu_r = 1 + \varkappa \ . \tag{1.9}$$

For anisotropic substances, χ, $\varkappa$, ε and μ take the form of symmetrical tensors of the second order.

In these so-called *soft* dielectric and magnetic mediums, the first series of Maxwell's equations takes the form

$$\mathrm{div}\boldsymbol{E} = \frac{\rho}{\varepsilon}, \qquad \mathrm{rot}\boldsymbol{B} - \varepsilon\mu\frac{\partial \boldsymbol{E}}{\partial t} = \mu \boldsymbol{j} \ , \tag{1.10}$$

where we still assumed that the medium is homogeneous and does not change with time (ε and μ are constant). In vacuum, the equations (1.10) will merge into the Lorentz-Maxwell equations 1.1.

Due to the electric and magnetic fields and also their names (intensity and induction), the apparent asymmetry of the definitions of the vectors ***D***, ***P*** and ***H***, ***M*** has a historical origin in the fact that the first theory of the magnetic interaction (permanent magnets) was based on the analogy with Coulomb's law for magnetostatics.

Maxwell's equations represent the pattern of a new kind of laws which are fundamentally different from the laws of mechanics, since they describe a new form of physical reality, the electromagnetic field.

The physical significance of Maxwell's equations

The physical meaning of the individual Maxwell's equations is evident from their integral form, which we will now derive. We will transform the equations of the first series of Maxwell's equations using mathematical theorems of vector analysis of Gauss and Stokes:

1st law: in a medium

$$\mathrm{div}\boldsymbol{E} = \frac{\rho}{\varepsilon_0} \iff \oint_{\partial V} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{f} = \frac{1}{\varepsilon_0} \int_V \rho \, \mathrm{d}V \ .$$

in vacuum

$$\mathrm{div}\boldsymbol{D} = \rho \iff \oint_{\partial V} \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{f} = \int_V \rho \, \mathrm{d}V,$$

The integral form of the equation is represented by the physical *Gauss's law* (C. F. Gauss 1839), which describes the emergence of an electric field induced by free electric charges. The integration takes place over a closed motionless ∂V surface that bounds the volume V.

2nd law: in a medium

$$\mathrm{rot}\boldsymbol{H} - \frac{\partial \boldsymbol{D}}{\partial t} = \boldsymbol{j} \iff \oint_{\partial f} \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} - \frac{\mathrm{d}}{\mathrm{d}t} \int_f \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{f} = \int_f \boldsymbol{j} \cdot \mathrm{d}\boldsymbol{f},$$

in vacuum

$$\mathrm{rot}\boldsymbol{B} - \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} = \mu_0 \boldsymbol{j} \quad \Longleftrightarrow \quad \oint_{\partial f} \boldsymbol{B}\cdot \mathrm{d}\boldsymbol{l} - \frac{1}{c^2}\frac{\mathrm{d}}{\mathrm{d}t}\int_f \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{f} = \mu_0 \int_f \boldsymbol{j}\cdot \mathrm{d}\boldsymbol{f} \ .$$

The integral form of the equations expresses the *Ampère's law* (André Marie Ampère (1775-1836) in 1827) of the generation of a magnetic field induced by free electric currents supplemented by Maxwell with the *displacement current* that is not related to the charge transfer. The integration takes place along the closed stationary curve ∂f delimiting the surface *f*.

Similarly, we will transform the equations of the 2nd series of Maxwell's equations that have the same form both in a medium and in vacuum:

3rd law:

$$\mathrm{div}\boldsymbol{B} = 0 \quad \Longleftrightarrow \quad \oint_{\partial V} \boldsymbol{B}\cdot \mathrm{d}\boldsymbol{f} = 0$$

expresses the fact that *there are no magnetic charges*, i.e. magnetic field lines are either closed or are extending from infinity to infinity.

4th law:

$$\mathrm{rot}\boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0 \quad \Longleftrightarrow \quad \oint_{\partial f} \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{l} + \frac{\mathrm{d}}{\mathrm{d}t}\int_f \boldsymbol{B}\cdot \mathrm{d}\boldsymbol{f} = 0$$

represents Maxwell's generalization of the *Faraday's law of electromagnetic induction* (M. Faraday 1831, J. C. Maxwell 1862). The time-varying magnetic field induces an electric field at point, regardless of whether there is a conductive loop to prove its existence.

1.2. Description of the point charge using Dirac δ-function

We describe the distribution of a charge in space using the charge density $\rho(\boldsymbol{r}, t)$. In macroscopic electrodynamics, we are satisfied with the simplification implying that the charge is distributed continuously. In fact, this means that we define

$$\rho_\Delta(\boldsymbol{r},\ t) \ = \lim_{\mathrm{d}V\to\Delta} \frac{\mathrm{d}Q}{\mathrm{d}V} \ , \tag{1.11}$$

where d*Q* is a charge in volume d*V* and Δ indicates the smallest volume we can divide in terms of experimental accuracy. Volume Δ must be small with respect to the body under consideration, but large enough to contain a large number of elementary charges. If volume Δ' has atomic dimensions, then

$$\int_\Delta \rho_{\Delta'}\ \mathrm{d}V \ = \ \Delta\rho_\Delta, \qquad \text{tj.} \quad \rho_\Delta \ = \ \frac{1}{\Delta}\int_\Delta \rho_{\Delta'}\ \mathrm{d}V$$

must be valid and applied as averaging due to limited experimental accuracy. Therefore, we can assume that ρ is a sufficiently smooth function. At small distances with respect to Δ, the exact course of the charge distribution no longer matters. It is often advisable to use the idealization of point charges. Let us demonstrate how it can be described using the Dirac δ-function.

What density $\rho(\boldsymbol{r}, t)$ should be assigned, e.g., to a point charge? For simplicity, let us assume the 'one-dimensional particle' with charge *Q* distributed so that $\rho(x) \neq 0$ for $x \in \langle x_1, x_2 \rangle$, $\rho(x) = 0$ for *x* not located in the interval $\langle x_1, x_2 \rangle$ and

$$Q \ = \int_{-\infty}^{+\infty} \rho(x)\ \mathrm{d}x \ .$$

At the same time, if we want to proceed to a point particle with a charge concentrated at point $x = x_0$, for density the following condition must be met

$$\int_{-\infty}^{+\infty} \rho(x)\ \mathrm{d}x \ = \ Q \quad \text{a} \quad \rho(x) \ = \ 0 \quad \text{při} \quad x \neq x_0 \ .$$

Functions with such properties do not exist in mathematical analysis. In 1930s, P. A. M. Dirac introduced a 'function' $\delta(x)$ with properties

$$\int_{-\infty}^{+\infty} \delta(x)\ \mathrm{d}x \ = \ 1 \quad \text{a} \quad \delta(x) \ = \ 0 \quad \text{při} \quad x \neq 0 \ ,$$

but the exact definition of this mathematical object was given only in 1950s by L. A. Schwartz in his theory of distribution (generalized functions).

Using the δ-function, we express the one-dimensional density of the point charge Q in point $x = 0$ as

$$\rho(x) \;=\; Q\;\delta(x)\;,$$

for the charge in point $x_0 = 0$ we have

$$\rho(x) \;=\; Q\;\delta(x \;-\; x_0).$$

For a three-dimensional particle at point r_0 we express

$$\rho(\boldsymbol{r}) \;=\; Q\;\delta^3(\boldsymbol{r} \;-\; \boldsymbol{r}_0),$$

where the three-dimensional δ-function is defined as a product

$$\delta^3(\boldsymbol{r} \;-\; \boldsymbol{r}_0) \;=\; \delta(x \;-\; x_0)\;\delta(y \;-\; y_0)\;\delta(z \;-\; z_0)$$

and meets the relations

$$\int_{R^3} \delta^3(\boldsymbol{r})\;\mathrm{d}V \;=\; 1 \qquad \mathrm{a} \qquad \delta^3(\boldsymbol{r}) \;=\; 0 \quad \mathrm{pro} \quad \boldsymbol{r} \;\neq\; \boldsymbol{0}.$$

Thus, the system N of point charges e_α at points r_a where $\alpha = 1,..., N$, will have the charge density

$$\rho(\boldsymbol{r}) \;=\; \sum_{\alpha=1}^{N} e_\alpha\;\delta^3(\boldsymbol{r} \;-\; \boldsymbol{r}_\alpha) \tag{1.12}$$

and current density

$$\boldsymbol{j}(\boldsymbol{r}) \;=\; \sum_{\alpha=1}^{N} e_\alpha\;\boldsymbol{v}_\alpha\;\delta^3(\boldsymbol{r} \;-\; \boldsymbol{r}_\alpha), \tag{1.13}$$

where the v_a are the charge velocities. These expressions should be substituted into the right sides of the first series of Lorentz-Maxwell equations 1.1.

Mathematical comment 1. Dirac δ-function

Let us introduce the basic rules for working with δ-functions:

$$\int_{-\infty}^{\infty} \delta(x)\;f(x)\mathrm{d}x = f(0),$$

$$\int_{-\infty}^{\infty} \delta(x - x_0)\;f(x)\mathrm{d}x = f(x_0),$$

$$\int_{R^3} \delta^3(\boldsymbol{r} - \boldsymbol{r}_0)\;f(\boldsymbol{r})\mathrm{d}V = f(\boldsymbol{r}_0),$$

$$\delta(-x) \;=\; \delta(x)\;,$$

$$x\;\delta(x) \;=\; 0\;,$$

$$\int_{-\infty}^{+\infty} \delta'(x \;-\; x_0)\;f(x)\;\mathrm{d}x \;=\; -f'(x_0)\;,$$

$$\int_{-\infty}^{+\infty} \delta^{(k)}(x \;-\; x_0)\;f(x)\;\mathrm{d}x \;=\; (-1)^k\;f^{(k)}(x_0)\;,\quad k = 1, 2, ...$$

$$\delta(c\,x) \;=\; \frac{1}{|c|}\,\delta(x), \quad c \neq 0 \,.$$

The last relation is a special case of identity

$$\delta[g(x)] \;=\; \sum_i \frac{1}{|g'(x_i)|}\,\delta(x \;-\; x_i)\;,$$

where g is a function differentiable at points x_i in which $g(x_i) = 0$.

Lastly, let us prove an important relation

$$\Delta\frac{1}{r} \;=\; -\,4\pi\;\delta^3(\boldsymbol{r})\;. \tag{1.15}$$

(Compare with the Poisson's equation of electrostatics.) In this case, the LaPlace operator Δ is reduced to the radial part

$$\Delta_r \;=\; \frac{1}{r^2}\,\frac{\mathrm{d}}{\mathrm{d}r}\,\left(r^2\,\frac{\mathrm{d}}{\mathrm{d}r}\right)\;,$$

and therefore

$$\Delta\frac{1}{r} \;=\; \Delta_r\frac{1}{r} \;=\; 0$$

at $\boldsymbol{r} \neq \boldsymbol{0}$. On the other side, for any area V containing $\boldsymbol{r} = \boldsymbol{0}$, we will receive a non-zero result using the Gauss's theorem

$$\int\limits_V \Delta\frac{1}{r}\,\mathrm{d}V \;=\; \int\limits_V \mathrm{div\ grad}\frac{1}{r}\,\mathrm{d}V \;=\; \oint\limits_{\partial V} \mathrm{grad}\frac{1}{r}\cdot\mathrm{d}\boldsymbol{f} \;=\; -\oint\limits_{\partial V}\frac{1}{r^2}\,\frac{\boldsymbol{r}}{r}\cdot\boldsymbol{n}\,\mathrm{d}f \;=\; -\int \mathrm{d}\Omega \;=\; -\,4\pi\;,$$

since dΩ is a spatial angle defined by the surface element d $\boldsymbol{f}$, and its vertex $\boldsymbol{r} = 0$ is assumed to be within the area V.

1.3 Electromagnetic potentials

In some special cases, Maxwell's equations can be solved directly as a system of partial differential equations of the first order for six unknown functions E_i, B_j. Since the theory of solving partial differential equations of the second order is elaborated better, it is useful to convert Maxwell's equations into an equivalent system of equations of this type.

This is done by the *method of potentials* which uses the fact that the second series of Maxwell's equations does not depend on the sources ρ, $\boldsymbol{j}$ and gives relations only for $\boldsymbol{E}$, $\boldsymbol{B}$. The equation div$\boldsymbol{B}$ = 0 can be easily fulfilled identically by introducing a *vector potential* $\boldsymbol{A}(\boldsymbol{r}, t)$ by relation

$$\boldsymbol{B} = \mathrm{rot}\boldsymbol{A}. \tag{1.16}$$

Then the first of the equations of the second series takes the form

$$\mathrm{rot}\left(\boldsymbol{E} + \frac{\partial\boldsymbol{A}}{\partial t}\right) = 0, \tag{1.17}$$

which we can easily fulfill identically by introducing the *scalar potential* φ ($\boldsymbol{r}$, t) by the relation

$$\boldsymbol{E} \;+\; \frac{\partial\boldsymbol{A}}{\partial t} \;=\; -\mathrm{grad}\;\varphi \tag{1.18}$$

(div rot $\boldsymbol{A}$ = 0, rot grad φ = 0 is valid). Thus, we found a general solution for the second series of Maxwell's equations

$$\boldsymbol{E} = -\mathrm{grad}\varphi - \frac{\partial\boldsymbol{A}}{\partial t}, \qquad \boldsymbol{B} = \mathrm{rot}\boldsymbol{A}. \tag{1.19}$$

Potentials φ, $\boldsymbol{A}$ (4 unknown functions) are arbitrary so far.

The equation for determining potentials is obtained from the first series of Maxwell's equations. We will further consider only a homogeneous isotropic medium where the equations of the first series are applied in the form (1.10), or vacuum. Then after substituting (1.19) to (1.10) and symmetrization, we will get a relatively complex set of differential equations of the second order for φ and $\boldsymbol{A}$:

$$\begin{aligned} \Delta\varphi - \varepsilon\mu\frac{\partial^2\varphi}{\partial t^2} &= -\frac{\rho}{\varepsilon} - \frac{\partial}{\partial t}\left(\mathrm{div}\boldsymbol{A} + \varepsilon\mu\frac{\partial\varphi}{\partial t}\right), \\ \Delta\boldsymbol{A} - \varepsilon\mu\frac{\partial^2\boldsymbol{A}}{\partial t^2} &= -\mu\boldsymbol{j} + \mathrm{grad}\left(\mathrm{div}\boldsymbol{A} + \varepsilon\mu\frac{\partial\varphi}{\partial t}\right) \end{aligned} \tag{1.20}$$

(rot rot$\boldsymbol{A}$ = grad div$\boldsymbol{A}$ — $\Delta\boldsymbol{A}$, Δ = div grad are valid).

In order to further simplify these equations, let us return to the definition of potentials (1.16), (1.18). The potential $\boldsymbol{A}$ is not clearly determined by relation $\boldsymbol{B}$ = rot$\boldsymbol{A}$ in a given field $\boldsymbol{B}$. Indeed, if we change $\boldsymbol{A}$ by the gradient of an arbitrary scalar function $\Lambda(\boldsymbol{r}, t)$

$$\boldsymbol{A} \longrightarrow \boldsymbol{A}' = \boldsymbol{A} + \mathrm{grad}\Lambda, \tag{1.21}$$

then, due to the identity of rot grad $\Lambda = 0$, the new vector potential $\boldsymbol{A}'$ describes the same magnetic field as $\boldsymbol{A}$. In order not to change the electric field $\boldsymbol{E}$ during the transformation (1.21), it is necessary to compensate for the change $\boldsymbol{A}$ by the change of φ. Using (1.21), the condition

$$\boldsymbol{E} = -\mathrm{grad}\varphi' - \frac{\partial\boldsymbol{A}'}{\partial t} = -\mathrm{grad}\varphi - \frac{\partial\boldsymbol{A}}{\partial t}$$

implies the relation

$$\mathrm{grad}\left(\varphi' - \varphi + \frac{\partial\Lambda}{\partial t}\right) = 0.$$

Since an arbitrary integration constant may be included in $\partial\Lambda/\partial t$, we can write the appropriate transformation of the scalar potential as

$$\varphi \longrightarrow \varphi' = \varphi - \frac{\partial\Lambda}{\partial t}. \tag{1.22}$$

The obtained transformations of potentials (1.21), (1.22) are referred to as *gauge* (also *calibration*) because they do not change measurable quantities $\boldsymbol{E}$ and $\boldsymbol{B}$. We call them *gauge invariant quantities.* Thus, we see that a whole class of pairs of potentials $\{(\varphi, \boldsymbol{A})\}$ united by gauge transformations corresponds to one of the electromagnetic fields $\boldsymbol{E}$, $\boldsymbol{B}$.

Equations (1.20) can now be simplified by appropriate selection of potentials from this class. For example, by applying the *Lorenz secondary condition*[3]

$$\mathrm{div}\boldsymbol{A} + \varepsilon\mu\frac{\partial\varphi}{\partial t} = 0 \tag{1.23}$$

we ensure that the equations (1.20) will no longer be bound:

$$\begin{aligned} \Delta\varphi - \varepsilon\mu\frac{\partial^2\varphi}{\partial t^2} &= -\frac{\rho}{\varepsilon}, \\ \Delta\boldsymbol{A} - \varepsilon\mu\frac{\partial^2\boldsymbol{A}}{\partial t^2} &= -\mu\boldsymbol{j}. \end{aligned} \tag{1.24}$$

The solution of Maxwell's equations is then transferred to an equivalent problem of inhomogeneous wave (d'Alembert's) equations (1.24) for the potentials φ, $\boldsymbol{A}$, that are found among those which meet Lorenz condition. The resulting field is then found from equations (1.19). In vacuum ($\varepsilon = \varepsilon_0$, $\mu = \mu_0$), d'Alembert's equations for potentials, together with Lorenz condition, take the form

$$\begin{aligned} \Box\ \varphi &\equiv \Delta\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -\frac{\rho}{\varepsilon_0} \\ \Box\ \boldsymbol{A} &\equiv \Delta\boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} = -\mu_0\boldsymbol{j} \\ &\mathrm{div}\boldsymbol{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t} = 0, \end{aligned} \tag{1.25}$$

[3]L. V. Lorenz introduced the aforementioned gauge in 1867, offering an alternative to the Coulomb gauge used by Maxwell. Lorenz gauge is frequently attributed to H. A. Lorentz, who is cited significantly more often in connection with the transformations of the electromagnetic field. Note that there is also Lorentz-Lorenz law (1869 and 1878, also referred to as Clausius-Mossotti law) that determines the refractive index of the medium depending on its polarizability.

where $\Box = \Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ is the d'Alembert's operator (d'Alembertian). These equations together with the equation (1.19) are then equivalent to the Lorentz-Maxwell equations 1.1.

Let us demonstrate more precisely what the *Lorenz gauge*, i.e. limiting potentials by the Lorenz condition (1.23) means. Let φ, ***A*** be some of the potentials that solve complex equations (1.20) but do not meet the Lorenz condition (1.23). Let us look for such potentials φ', ***A***' in the class of gauge equivalent potentials to φ, ***A***, for which Lorenz's condition is applied

$$\mathrm{div}\boldsymbol{A}' + \varepsilon\mu\frac{\partial\varphi'}{\partial t} = 0. \tag{1.26}$$

If we substitute here for A' and φ', we get the equation

$$\Delta\Lambda - \varepsilon\mu\frac{\partial^2\Lambda}{\partial t^2} = -\left(\mathrm{div}\boldsymbol{A} + \varepsilon\mu\frac{\partial\varphi}{\partial t}\right), \tag{1.27}$$

which determines the unknown function Λ (***r***, *t*) at the prescribed (non-zero) right side. As we shall see in the following chapter, such an inhomogeneous wave equation has (infinitely many) solutions, and therefore, the potentials φ', ***A***' fulfilling Lorenz condition (1.26) do exist. The system (1.20) is then simplified to the system (1.24) for unknown φ', ***A***'.

Furthermore, let us also make sure that the time evolution of the potentials φ, ***A*** according to (1.24) does not break the Lorenz condition (1.23), i.e. if (1.23) is fulfilled at time t_0, it will also apply to all $t > t_0$. To do this, it is sufficient to apply the operators $\varepsilon\mu\, \partial/\partial t$ and div to the equations (1.24), to sum up the resulting equations and reverse the order of derivations. After the transformation, we will receive the equation

$$\Delta\left(\mathrm{div}\boldsymbol{A} + \varepsilon\mu\frac{\partial\varphi}{\partial t}\right) - \varepsilon\mu\frac{\partial^2}{\partial t^2}\left(\mathrm{div}\boldsymbol{A} + \varepsilon\mu\frac{\partial\varphi}{\partial t}\right) = -\mu\left(\mathrm{div}\boldsymbol{j} + \frac{\partial\rho}{\partial t}\right),$$

whose right side is zero due to the equation of continuity. Thus, the zero solution of this equation means meeting the Lorenz condition at any time.

Let us return to the fact that the solution to the equation (1.27) is not determined uniquely: if we have one (particular) solution Λ, the next one is obtained by adding any solution Λ' to a homogeneous equation

$$\Delta\Lambda' - \varepsilon\mu\frac{\partial^2\Lambda'}{\partial t^2} = 0. \tag{1.28}$$

There still remains a lot of ambiguity in the choice of potentials that can be used to further specify the gauge fixing. The resulting gauge transformation will be determined by the sum of Λ + Λ':

$$\varphi'' = \varphi' - \frac{\partial\Lambda'}{\partial t} = \varphi - \frac{\partial}{\partial t}(\Lambda + \Lambda'), \tag{1.29}$$

$$\boldsymbol{A}'' = \boldsymbol{A}' + \mathrm{grad}\ \Lambda' = \boldsymbol{A} + \mathrm{grad}\ (\Lambda + \Lambda'). \tag{1.30}$$

For example, when $\rho = 0$, we may require additional conditions to be met

$$\varphi'' = 0.$$

We see that this condition corresponds to such a solution of the wave equation Λ' (1.28), which according to (1.29) fulfills the equation

$$\frac{\partial\Lambda'}{\partial t} = \varphi'. \tag{1.31}$$

This equation is apparently fulfilled by the function in the form $\Lambda'(\boldsymbol{r}, t) = \int_0^t \varphi'(\boldsymbol{r}, t')\mathrm{d}t' + \beta(\boldsymbol{r})$. According to the wave equation (1.28), $\beta(\boldsymbol{r})$ will then be the solution of Poisson equation $\Delta\beta(\boldsymbol{r}) = \varepsilon\mu(\partial\varphi'/\partial t)\ (\boldsymbol{r}, 0)$. This is easily demonstrated using the wave equation for φ' in the area of space where $\rho = 0$. Then instead of the system (6.33), (1.24), in the area of space, where $\rho = 0$, we will have the equation

$$\mathrm{div}\boldsymbol{A} = 0, \quad \varphi = 0, \quad \Delta\boldsymbol{A} - \varepsilon\mu\frac{\partial^2\boldsymbol{A}}{\partial t^2} = -\mu\boldsymbol{j}. \tag{1.32}$$

This gauge fixing is a special case of the so-called *Coulomb gauge* for $\rho = 0$.

1.4 Laws of conservation of charge, energy and momentum

In this paragraph, we will derive the integral and differential relations expressing the laws of conservation of charge, the energy of the electromagnetic field and the three components of its momentum.

The law of conservation of charge

Electric charge is not a separate substance, but is carried by some material particles, such as electrons, protons or atomic nuclei. Its most important feature is the *law of charge conservation.* In the field of energies that are low with respect to the rest energy of the charged particles this law is a consequence of law of conservation of the number of particles, which results from the basic equations of quantum mechanics. It has been verified that the law also applies to elementary quantum processes in which particles originate and disappear; let us mention e.g. the beta decay of the neutron $n \rightarrow p + e^- + \tilde{\nu}_e$ or annihilation of the electron-positron pair $e^- + e^+ \rightarrow 2\gamma$. In all these processes, the *sum* of the charges is preserved; therefore, we are speaking of the *additive law of conservation.* The first experimental proof of this law was given by Michael Faraday in 1843 (See also [22]).

However, the charge has other properties, e.g. it is quantized and, unlike mass, it is relativistically invariant, it does not change when the particle moves. The charge is a scalar and can have a positive or negative sign. After the discovery of anti-particles, the charge symmetry of particles and anti-particles turned out to play an important role in the universe and to be related to other spatial and quantum symmetries. However, it took a century to clarify whether electrical phenomena are the manifestation of only one type of charge ('single-fluid theory') or whether there are two oppositely charged electrical 'fluids'.

The law of conservation of a charge contained in a given arbitrary spatial region V bounded by the surface ∂V can be expressed by equality: *the rate of loss of charge in the volume V is equal to the amount of charge that passes through the stationary surface ∂V per unit of time.* Mathematically, the equality reads

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho \,\mathrm{d}V = \oint_{\partial V} \boldsymbol{j}\cdot\mathrm{d}\boldsymbol{f},$$

and represents an integral form of the equation of electrical current continuity. According to Gauss's theorem, here we obtain a differential continuity equation in the limit $V \rightarrow 0$

$$\frac{\partial\rho}{\partial t} + \operatorname{div}\boldsymbol{j} = 0, \tag{1.33}$$

which expresses *the local law of charge conservation.*

It is evident that the equation of continuity is a consequence of the first series of Maxwell's equations (it is sufficient to use the operators $\partial/\partial t$ and div for the equations (1.5) and sum up the resulting equations.) However, the physical meaning of the continuity equation, whose validity was demonstrated outside the framework of Maxwell's equations, illustrates the following point of view in a better way. Maxwell's equations

$$\operatorname{rot}\boldsymbol{E} + \frac{\partial\boldsymbol{B}}{\partial t} = 0, \qquad \operatorname{rot}\boldsymbol{H} - \frac{\partial\boldsymbol{D}}{\partial t} = \boldsymbol{j}$$

describe the time development of the electromagnetic field. If we apply div operator for both of them and use the expression (1.33), we will obtain

$$\frac{\partial}{\partial t}\operatorname{div}\boldsymbol{B} = 0, \qquad \frac{\partial}{\partial t}(\operatorname{div}\boldsymbol{D} - \rho) = 0.$$

We can interpret these equations in such a way that due to the validity of the equation continuity, the equation

$$\operatorname{div}\boldsymbol{B} = 0, \qquad \operatorname{div}\boldsymbol{D} = \rho \tag{1.34}$$

is valid at any time t, if at a certain time $t = t_0$. Thus, from this standpoint, the expressions (1.34) can be considered as the initial conditions.

Energy Conservation Law

According to all experience to date, physical phenomena are in accordance with the laws of conservation of energy and momentum. To derive the *energy balance* in the electromagnetic field, let us consider the system of charged particles and the electromagnetic field. The field energy in the spatial region V bounded by a *stationary closed surface ∂V, which particles do not pass through,* can only be consumed by the work W performed by the field while interacting with the particles in the volume V and by the energy flow through the boundary ∂V of the region V into the outer space.

If the charges have density distribution $\rho(\boldsymbol{r}, t)$ and the velocity distribution $v(\boldsymbol{r}, t)$, then the power of the field force dW/dt in the region V will be determined by the Lorentz force density.

$$\boldsymbol{f} = \rho\ (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) = \rho\ \boldsymbol{E} + \boldsymbol{j} \times \boldsymbol{B} \tag{1.35}$$

by relation

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \int\limits_V \boldsymbol{f} \cdot \boldsymbol{v}\ \mathrm{d}V = \int\limits_V \rho\ \boldsymbol{v} \cdot \boldsymbol{E}\ \mathrm{d}V = \int\limits_V \boldsymbol{j} \cdot \boldsymbol{E}\mathrm{d}V \tag{1.36}$$

(Note that $\boldsymbol{f}$ dV is a force acting on the charge ρ dV) Thus, the quantity $\boldsymbol{j} \cdot \boldsymbol{E}$ represents *the density of the mechanical power of the forces of the field*. This quantity induces the Joule heating generated in a unit of volume per second in the conductor.

If w denotes the field energy density and $\boldsymbol{S}$ denotes the field energy flux density, we can express the energy balance by equation

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_V w\ \mathrm{d}V = \int\limits_V \boldsymbol{j} \cdot \boldsymbol{E}\ \mathrm{d}V + \oint\limits_{\partial V} \boldsymbol{S} \cdot \mathrm{d}\boldsymbol{f}. \tag{1.37}$$

As in the case of the continuity equation, we can arrive from this integral form into a differential form

$$-\frac{\partial w}{\partial t} = \boldsymbol{j} \cdot \boldsymbol{E} + \mathrm{div}\ \boldsymbol{S}, \tag{1.38}$$

which expresses *the local energy conservation law in the system of electromagnetic field and charged particles.*

To determine how w and $\boldsymbol{S}$ depend on the electromagnetic field, we will transform the expression $\boldsymbol{j} \cdot \boldsymbol{E}$ using Maxwell's equations (1.5) and identity

$$\mathrm{div}\ (\boldsymbol{E} \times \boldsymbol{H}) = \boldsymbol{H} \cdot \mathrm{rot}\ \boldsymbol{E} - \boldsymbol{E} \cdot \mathrm{rot}\ \boldsymbol{H}\ \ . \tag{1.39}$$

Gradually we will obtain

$$\boldsymbol{j} \cdot \boldsymbol{E} = \left(\mathrm{rot}\ \boldsymbol{H} - \frac{\partial \boldsymbol{D}}{\partial t}\right) \cdot \boldsymbol{E} = -\ \mathrm{div}\ (\boldsymbol{E} \times \boldsymbol{H}) + \left(\boldsymbol{H} \cdot \mathrm{rot}\ \boldsymbol{E} - \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t}\right) = -\mathrm{div}\ (\boldsymbol{E} \times \boldsymbol{H}) - \left(\boldsymbol{H} \cdot \frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t}\right)\ \ .\ \ .$$

For a soft medium, whose permittivity ε and permeability μ do not depend on time, the *Poynting theorem* in a simple and commonly used form [10, 23] is valid

$$-\ \frac{\partial}{\partial t}\ \frac{1}{2}\ (\boldsymbol{E} \cdot \boldsymbol{D}\ +\ \boldsymbol{H} \cdot \boldsymbol{B})\ =\ \boldsymbol{j} \cdot \boldsymbol{E}\ +\ \mathrm{div}\ (\boldsymbol{E} \times \boldsymbol{H})\ \ . \tag{1.40}$$

For the field energy density, the comparison of the expression (1.40) with (1.38) here gives

$$w\ =\ \frac{1}{2}\ (\boldsymbol{E} \cdot \boldsymbol{D}\ +\ \boldsymbol{H} \cdot \boldsymbol{B})\ =\ \frac{1}{2}\ (\varepsilon \boldsymbol{E}^2\ +\ \mu \boldsymbol{H}^2)$$

and for $\boldsymbol{S}$ *(Poynting vector*)

$$\boldsymbol{S}\ =\ \boldsymbol{E} \times \boldsymbol{H}.$$

The energy density of the field in vacuum is

$$w\ =\ \frac{1}{2}\ (\varepsilon_0 \boldsymbol{E}^2\ +\ \mu_0 \boldsymbol{H}^2),$$

where $\boldsymbol{H} = \boldsymbol{B}/\mu_0$. In a material medium, the formula (1.41) includes, in addition to vacuum field energy, the energy consumed by the polarization and magnetization of the medium.

The law of momentum conservation

We have seen that the electromagnetic field acts on the charge and current carriers by the force with volume density (1.35), which, according to (1.36) is changing their energy. However, we know that according to the law of force, their momentum will also change. The experimental evidence of the existence of the momentum in the electromagnetic field was brought by the discovery of mechanical pressure of radiation (Pyotr Nikolaevich Lebedev (1866-1912) in 1899).

Let us write the balance of momentum in the system of electromagnetic field and charged particles: the loss of momentum of this system in the volume V bounded by a *motionless surface ∂V that particles do not pass through*, is equal to the momentum of the electromagnetic field, which flows from the volume V through the boundary ∂V. If we denote the particle momentum density as $\boldsymbol{p}$, $\boldsymbol{g}$ is the field momentum density and (σ_{i1}, σ_{i2}, σ_{i3}) is the flux density vector of the field momentum (the flux density of the vector is a tensor), we can express the momentum balance in an integral form

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_V (p_i + g_i)\,\mathrm{d}V = \oint_{\partial V} \sigma_{ik}\,\mathrm{d}f_k\,, \quad (i=1,2,3). \tag{1.44}$$

The differential form of these equations

$$-\frac{\partial g_i}{\partial t} = \frac{\partial p_i}{\partial t} + \frac{\partial \sigma_{ik}}{\partial x_k}, \tag{1.45}$$

where $\partial\sigma_{ik}/\partial x_k$ is the tensor divergence, is the mathematical expression *of the local law of momentum conservation in the system of electromagnetic field and charged particles*.

In order to explicitly determine the quantities $\boldsymbol{g}$ and (σ_{ik}), we will write the equation of motion for particles in the volume unit,

$$\frac{\partial \boldsymbol{p}}{\partial t} = \boldsymbol{f} = \rho\,\boldsymbol{E} + \boldsymbol{j}\times\boldsymbol{B} \tag{1.46}$$

and we will express the densities ρ, $\boldsymbol{j}$ using Maxwell's equations (1.5):

$$\boldsymbol{f} = \boldsymbol{E}\,\mathrm{div}\,\boldsymbol{D} + \left(\mathrm{rot}\,\boldsymbol{H} - \frac{\partial \boldsymbol{D}}{\partial t}\right)\times\boldsymbol{B}\,.$$

If we add terms

$$\boldsymbol{H}\,\mathrm{div}\,\boldsymbol{B} = 0\,, \qquad \left(\mathrm{rot}\,\boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t}\right)\times\boldsymbol{D} = 0\,,$$

which are zero due to the second series of Maxwell's equations, to the right side, we will obtain a symmetrical expression

$$\boldsymbol{f} = -\frac{\partial}{\partial t}(\boldsymbol{D}\times\boldsymbol{B}) + [\boldsymbol{E}\,\mathrm{div}\,\boldsymbol{D} + \boldsymbol{H}\,\mathrm{div}\,\boldsymbol{B} - \boldsymbol{D}\times\mathrm{rot}\,\boldsymbol{E} - \boldsymbol{B}\times\mathrm{rot}\,\boldsymbol{H}\,]\,. \tag{1.47}$$

This equation is identical with the equation (1.45), if we manage to modify the vector in square brackets to the tensor divergence $-\sigma_{ik}$. Complex algebraic manipulations are made only for the first component:

$$[\ldots]_1 = E_1\frac{\partial D_1}{\partial x_1} + E_1\frac{\partial D_2}{\partial x_2} + E_1\frac{\partial D_3}{\partial x_3} -$$
$$-D_2\left(\frac{\partial E_2}{\partial x_1} - \frac{\partial E_1}{\partial x_2}\right) + D_3\left(\frac{\partial E_1}{\partial x_3} - \frac{\partial E_3}{\partial x_1}\right) + [\boldsymbol{E}\to\boldsymbol{H}\;,\quad \boldsymbol{D}\to\boldsymbol{B}]\,.$$

The last bracket denotes other terms of the same form in the magnetic quantities H, B. For a homogeneous soft medium

$$E_1\,\frac{\partial D_1}{\partial x_1} = \frac{1}{2}\,\frac{\partial}{\partial x_1}\,(E_1\,D_1), \quad E_1\,\frac{\partial D_2}{\partial x_2} = \frac{\partial}{\partial x_2}\,(E_1\,D_2) - \frac{\partial E_1}{\partial x_2}\,D_2,$$

and thus, after the interruption and regrouping the terms

$$[\ldots]_1 = \frac{\partial}{\partial x_1}(E_1\,D_1) + \frac{\partial}{\partial x_2}(E_1D_2) + \frac{\partial}{\partial x_3}(E_1D_3) - \frac{1}{2}\frac{\partial}{\partial x_1}(E_1D_1 + E_2D_2 + E_3D_3) + [\boldsymbol{E}\to\boldsymbol{H}, \boldsymbol{D}\to\boldsymbol{B}].$$

Based on this result, we can now write the i-th component of the equation (1.47) as

$$f_i = -\frac{\partial}{\partial t}(\boldsymbol{D}\times\boldsymbol{B})_i + \frac{\partial}{\partial x_k}\left\{E_iD_k + H_iB_k - \frac{1}{2}\delta_{ik}(\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{H}\cdot\boldsymbol{B})\right\}. \tag{1.48}$$

Comparison with relations (1.45) and (1.46) gives the following relation for the field momentum density

$$\boldsymbol{g} = \boldsymbol{D}\times\boldsymbol{B} = \varepsilon\,\mu\,\boldsymbol{E}\times\boldsymbol{H} \tag{1.49}$$

and for the tensor (σ_{ik}) it gives the expression

$$\sigma_{ik} = -\left[E_i\,D_k + H_i\,B_k - \frac{1}{2}\delta_{ik}\,(\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{H}\cdot\boldsymbol{B})\right]. \tag{1.50}$$

In *vacuum*

$$g = \varepsilon_0 \, \mu_0 \, \boldsymbol{E} \times \boldsymbol{H} = \frac{1}{c^2} \, \boldsymbol{S}, \tag{1.51}$$

$$\sigma_{ik} = - \left[\varepsilon_0 \, E_i \, E_k + \mu_0 H_i \, H_k - \frac{1}{2} \, \delta_{ik} \left(\varepsilon_0 \, \boldsymbol{E}^2 + \mu_0 \, \boldsymbol{H}^2 \right) \right] . \tag{1.52}$$

Since in the stationary field ∂g/∂t = 0 and the relation (1.45) transfers into

$$f_i = - \frac{\partial \sigma_{ik}}{\partial x_k} \, , \tag{1.53}$$

i.e., to an equation of the same form as the continuum statics equation (3.277), σ_{ik} is referred to as *Maxwell stress tensor* It allows to calculate the force acting on the volume *V* of substance as in the continuum mechanics, i.e. as a resultant of 'stresses' acting on the surface of the ∂*V*,

$$\int_V f_i \, \mathrm{d}V = \oint_{\partial V} (-\sigma_{ik}) \, \mathrm{d}f_k \, . \tag{1.54}$$

Note that the derived relations (1.49) and (1.50) have been discussed from the very beginning, because it is unclear how the momentum of bound charges in the material medium is included into them in addition to the actual momentum of the field. However, the correct momentum density of the electromagnetic field itself is given by the expression (1.51), which we would derive not from the average macroscopic field, but from the microscopic field. Maxwell stress tensor is also suitable for a field in soft medium (1.50). Detailed relations for different types of media can be found in the recent work[4]. According to this work, the relation (1.51) is valid not only in vacuum, but in a non-magnetic medium (***M*** = 0) independent of its dielectric properties. The general theory, however, suggests that the electromagnetic field also transmits angular momentum in addition to energy and momentum.

1.5 THE EQUATIONS OF ELECTRODYNAMICS IN MINKOWSKI SPACE-TIME

According to Einstein's principle of relativity, the laws of physics have the same form in all inertial systems. Mathematically, the relativistic invariance of physical laws can be expressed by such notations of the relevant equations, where all their terms are clearly of the same transformational properties as in Lorentz transformations. Such equations are referred to as *covariant.* If they are valid in one inertial system, they are valid in the same form in every other inertial system. In this sense, e.g. vector equations of mechanics are covariant with respect to rotations of the coordinate system.

Lorentz-Maxwell equations can also be written in an apparent covariant form, if we determine the correct transformation properties of all the quantities involved. Let us start from the charge we know to be one of the few relativistic invariants. The electric charge density is no longer an invariant, but it is transformed as a zero component of the four-vector, that is, e.g., time, because

$$\frac{\rho}{\mathrm{d}t} = \frac{\rho \, \mathrm{dV}}{\mathrm{d}t \, \mathrm{d}V} = c \, \frac{\mathrm{d}e}{\mathrm{d}V*} = \text{invariant} \, .$$

Based on this property, a four-vector can be defined

$$j^\mu = \rho \, \frac{\mathrm{d}x^\mu}{\mathrm{d}t} \tag{1.55}$$

with components

$$(j^\mu) = (c \, \rho, \, \rho \, \boldsymbol{v}) = (c \, \rho, \, \boldsymbol{j}) \, . \tag{1.56}$$

Due to the physical nature of its spatial part, we call it a *four-current.* It allows to give the continuity equation (1.33) a clearly covariant form of zero four-divergence

$$\frac{\partial j^\mu}{\partial x^\mu} = 0 \, . \tag{1.57}$$

[4]Shevchenko A., Kaivola M.: Electromagnetic force density and energy-momentum tensor in an arbitrary continuous medium, J. Phys. B: At. Mol. Opt. Phys. 44 (2011) 175401 (7pp)

We know that Lorentz-Maxwell equations are equivalent to a system of equations (1.25) for potentials defined by relations (1.19). Note that the right sides of the d'Alembert equations (1.25) are proportional to the four-vector components while the invariant d'Alembert operator acts on the left side

$$\Box = -g^{\mu\nu} \frac{\partial^2}{\partial x^\mu \, \partial x^\nu}.$$

Therefore, we can consider the scalar and vector potential as the time and space part of the *four-potential*

$$(A^\mu) = \left(\frac{\varphi}{c}, \boldsymbol{A}\right). \tag{1.58}$$

With this definition, equations (1.25) turn into the covariant d'Alembert equation

$$\Box A^\mu = -\mu_0 \, j^\mu \tag{1.59}$$

and Lorentz-invariant condition

$$\frac{\partial A^\mu}{\partial x^\mu} = 0. \tag{1.60}$$

The gauge transformations (1.21) and (1.22) can also be written in a covariant form

$$A^\mu \longrightarrow A'^\mu = A^\mu - \frac{\partial \Lambda}{\partial x_\mu}. \tag{1.61}$$

We will now express fields ***E***, ***B*** according to (1.19) using the four-potential components

$$\frac{1}{c} E_x = -\frac{\partial}{\partial x^1}\left(\frac{\varphi}{c}\right) - \frac{\partial A^1}{\partial (ct)} = \frac{\partial A_1}{\partial x^0} - \frac{\partial A_0}{\partial x^1}, \ldots,$$

$$B_x = \frac{\partial A^3}{\partial x^2} - \frac{\partial A^2}{\partial x^3} = \frac{\partial A_2}{\partial x^3} - \frac{\partial A_3}{\partial x^2}, \ldots .$$

Thus, they form six independent components of antisymmetric tensor of the second order in four-dimensional space, i.e. *electromagnetic field tensor*

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} . \tag{1.62}$$

(Verify that the tensor $F_{\mu\nu}$ is gauge invariant.) Explicitly

$$(F_{\mu\nu}) = \begin{pmatrix} 0, & \frac{E_x}{c}, & \frac{E_y}{c}, & \frac{E_z}{c} \\ -\frac{E_x}{c}, & 0, & -B_z, & B_y \\ -\frac{E_y}{c}, & B_z, & 0, & -B_x \\ -\frac{E_z}{c}, & -B_y, & B_x, & 0 \end{pmatrix} \tag{1.63}$$

and after raising both indices

$$(F^{\mu\nu}) = \begin{pmatrix} 0, & -\frac{E_x}{c}, & -\frac{E_y}{c}, & -\frac{E_z}{c} \\ \frac{E_x}{c}, & 0, & -B_z, & B_y \\ \frac{E_y}{c}, & B_z, & 0, & -B_x \\ \frac{E_z}{c}, & -B_y, & B_x, & 0 \end{pmatrix} \tag{1.64}$$

The relevant dual tensor

$$F^*_{\varkappa\lambda} = \frac{1}{2!} \varepsilon_{\varkappa\lambda\mu\nu} F^{\mu\nu}$$

is again an antisymmetric tensor of the second order. It has components

$$(F^{*\mu\nu}) = \begin{pmatrix} 0, & B_x, & B_y, & B_z \\ -B_x, & 0, & -\frac{E_z}{c}, & \frac{E_y}{c} \\ -B_y, & \frac{E_z}{c}, & 0, & \frac{E_x}{c} \\ -B_z, & -\frac{E_y}{c}, & \frac{E_x}{c}, & 0 \end{pmatrix}. \tag{1.65}$$

The Lorentz-Maxwell equations 1.1 can now be written in a compact and relativistically covariant form

$$\text{1st series:}\quad \frac{\partial F^{\mu\nu}}{\partial x^{\nu}} = -\mu_0\, j^{\mu}\,, \qquad \textit{and} \qquad \text{2nd series:}\quad \frac{\partial F^{*\mu\nu}}{\partial x^{\nu}} = 0\,. \tag{1.66}$$

It is apparent from the previous relations that the transformation of the duality $F \rightarrow F^*$ corresponds to the current substitution of $\boldsymbol{E}/c \rightarrow -\boldsymbol{B}$, $\boldsymbol{B} \rightarrow \boldsymbol{E}/c$, in which the left sides of the first series of the Lorentz-Maxwell equations will transfer (except for the proportionality constants) to the left side of the second series. Similarly, the dual tensor for the four-rotation (1.62) is the solution of the second series (1.66) with the use of potentials, and the first series is equivalent to the d'Alembert equation (1.59) with the Lorentz condition (1.60) .

The Lorentz force (1.3) or rather Lorentz force density (1.35) also has a simple covariant form. We will express the first component of the Lorentz four-force

$$K^1 = \frac{F_x}{\sqrt{1-\frac{v^2}{c^2}}} = e\,\frac{c}{\sqrt{1-\frac{v^2}{c^2}}}\,\frac{E_x}{c} + e\,\frac{v_y}{\sqrt{1-\frac{v^2}{c^2}}}\,B_z - e\,\frac{v_z}{\sqrt{1-\frac{v^2}{c^2}}}\,B_y$$

using the four-velocity components u^{μ} and tensor $F^{\mu\nu}$ as

$$K^1 = e\,F^{1\nu}\,u_{\nu}.$$

Similar relations are applied to all the components of Lorentz four-force

$$K^{\mu} = e\,F^{\mu\nu}\,u_{\nu}\,, \qquad (K^{\mu}) = \left(\frac{e\boldsymbol{E}\cdot\boldsymbol{v}}{c\sqrt{1-\frac{v^2}{c^2}}}, \frac{e(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})}{\sqrt{1-\frac{v^2}{c^2}}}\right), \tag{1.67}$$

where the zero component is proportional to the power of Lorentz force. Similar relations are also valid for covariant Lorentz force density:

$$f^{\mu} = F^{\mu\nu} j_{\nu}\,, \qquad (f^{\mu}) = \left(\frac{1}{c}\,\boldsymbol{E}\cdot\boldsymbol{j},\ \ \rho\,\boldsymbol{E}+\boldsymbol{j}\times\boldsymbol{B}\right). \tag{1.68}$$

Therefore, relativistic equation of motion (1.4) for a particle of rest mass mo and charge e in a given external electromagnetic field $F^{\mu\nu}$ will have a covariant form in Minkowski's four-space

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(m_0\,u^{\mu}) = e\,F^{\mu\nu}\,u_{\nu}. \tag{1.69}$$

1.6. Motion of the charged particle in external electric and magnetic field

Let us consider a particle of rest mass m_0 and positive charge q, which at time $t = 0$ moves from the origin O in the direction of the x-axis with a relativistic initial velocity $\boldsymbol{v}_0$. A *homogeneous electrostatic field* $\boldsymbol{E}$ acts in the direction of axis y. We shall determine the form of the particle orbit.

Let us write the relativistic equation of motion of a particle in the form

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = q\,\boldsymbol{E} \tag{1.70}$$

(the charge of the particle is relativistically invariant). By integrating with the given initial conditions, we will obtain

$$p_x = p_0 = \text{konst.}\,, \quad p_y = q\,E\,t\,. \tag{1.71}$$

Now we must take into account the fact that the relativistic relation between the momentum and velocity is

$$\boldsymbol{v} = \frac{\boldsymbol{p}\,c^2}{\mathcal{E}}\,, \quad \text{kde} \quad \mathcal{E} = c\,\sqrt{p^2 + m_0^2\,c^2}\,. \tag{1.72}$$

To avoid confusion with the electric field, we shall denote the energy of the free particle as $\mathcal{E}$. If we mark the initial value of energy

$$\mathcal{E}_0 = c\,\sqrt{p_0^2 + m_0^2\,c^2}\,, \tag{1.73}$$

we can write

$$\mathcal{E} = c\,\sqrt{p_0^2 + q^2\,E^2\,t^2 + m_0^2\,c^2} = \sqrt{\mathcal{E}_0^2 + (c\,q\,E\,t)^2}\,. \tag{1.74}$$

We therefore have

$$v = \frac{p\, c^2}{\sqrt{\mathcal{E}_0^2 + (c\, q\, E\, t)^2}} , \tag{1.75}$$

and in the components

$$v_x = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{c^2\, p_0}{\sqrt{\mathcal{E}_0^2 + (c\, q\, E\, t)^2}} , \quad v_y = \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{c^2\, q\, E\, t}{\sqrt{\mathcal{E}_0^2 + (c\, q\, E\, t)^2}} . \tag{1.76}$$

By further integration, we will receive[5]

$$x = \frac{c\, p_0}{q\, E} \operatorname{arg\,sinh} \frac{c\, q\, E\, t}{\mathcal{E}_0} , \quad y = \frac{1}{q\, E} \sqrt{\mathcal{E}_0^2 + (c\, q\, E\, t)^2} - \frac{\mathcal{E}_0}{q\, E} . \tag{1.77}$$

The equations (1.77) indicate the law of motion and represent the parametric equations of the trajectory in the surface *x, y*. If we exclude time from them, we will obtain the equation of the particle's orbit

$$y = \frac{\mathcal{E}_0}{q\, E} \left(\cosh \frac{q\, E\, x}{c\, p_0} - 1 \right) . \tag{1.78}$$

Thus, the motion of the particle follows a curve referred to as *catenary*. The constants $\mathcal{E}_0$ and p_0 o are bound by the relation (1.73).

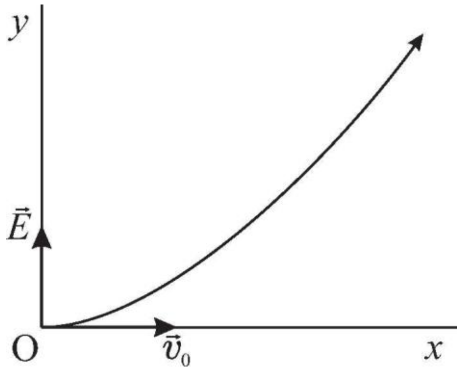


Figure 1.1: A relativistic charged particle in a constant electric field

At non-relativistic velocities, we have $\mathcal{E}_0 = m_0 c^2$, $p_0 = m_0 v_0$ and we will expand the hyperbolic cosine according to a small argument

$$\cosh x = \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2} = 1 + \frac{x^2}{2!} + \ldots .$$

Then the expression (1.78) transforms into a non-relativistic parabola

$$y = \frac{q\, E}{2\, m_0\, v_0^2}\, x^2 . \tag{1.79}$$

Now let us consider the case where a particle of rest mass m_0, charge *q* and relativistic initial velocity v is moving in a *homogeneous magnetic field **B***.

We shall write the equation of motion of a particle in the form

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = q\, \boldsymbol{v} \times \boldsymbol{B} . \tag{1.80}$$

If we use the relation between the relativistic velocity and momentum of a particle p = $(\mathcal{E}/c^2)\, \boldsymbol{v}$, we can rewrite (1.80) as

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{q\, c^2}{\mathcal{E}}\, \boldsymbol{v} \times \boldsymbol{B} . \tag{1.81}$$

[5]arg sinh *x* is an inverse function to the hyperbolic sinus and arg sinh *(x/a)* = ln|x + $\sqrt{x^2 + a^2}$| is valid.

Since the magnetic field does not perform work on a particle (Lorentz force remains perpendicular to velocity), the energy $\mathcal{E}$ is constant during the motion.

Let us remind that in a non-relativistic case, we solve the equation of motion

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{q}{m_0}\,\boldsymbol{v}\times\boldsymbol{B}\,, \tag{1.82}$$

arrive at the result that the particle performs a helical motion with an axis parallel to the magnetic field (see [22] p. 356). In a plane perpendicular to the magnetic field, this motion appears to be a uniform circular motion with an angular velocity $\omega_c = qB/m_0$ and a radius $r_L = m_0v_0t/qB$. In the direction of the magnetic field, the particle performs a uniform motion at a speed v_{0l}; v_{0t} and v_{0l} are the components of the initial velocity in the perpendicular and longitudinal direction with respect to the magnetic field, where $v_{0t} = v_t$, $v_{0l} = v_l$, $v_0 = v$ and they do not change during the motion.

The relativistic and non-relativistic equations of motion (1.81) and (1.82) differ only by the constant before the product $\boldsymbol{v}\times\boldsymbol{B}$. Using the same procedure as in the solution of the non-relativistic problem, we will come to the conclusion that even relativistic particles will move along the helix. The cyclotron angular frequency and the Larmor radius will equal

$$\omega_{cr} = \frac{q\,c^2\,B}{\mathcal{E}} = \frac{q\,B}{m_0}\sqrt{1-\frac{v^2}{c^2}}\,,\qquad r_{Lr} = \frac{v_{0t}}{\omega_{cr}} = \frac{v_{0t}\,\mathcal{E}}{q\,c^2\,B} = \frac{m_0\,v_{0t}}{q\,B}\,\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\,. \tag{1.83}$$

If the initial velocity of the particle $v_0 = v$ increases, the cyclotron frequency ω_{cr} decreases and the Larmor radius r_{Lr} increases.

1.7 Lorentz transformation of potentials and fields, invariants

We derive transformation formulas for potentials φ, $\boldsymbol{A}$ and field $\boldsymbol{E}$, $\boldsymbol{B}$ in a special Lorentz transformation $x'^\mu = \alpha^\mu_{.\nu}x^\nu$. The four-potential (A^μ) _ $(\varphi/c, \boldsymbol{A})$ is transformed as a four-vector

$$A'^\mu(x') = \alpha^\mu_{.\nu}\,A^\nu(x). \tag{1.84}$$

The matrix of the special Lorentz transformation is

$$(\alpha^\mu{}_\nu) = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\beta = V/c$, $\gamma = 1/\sqrt{1-\beta^2}$. It is then valid that

$$A'^0 = \gamma\,(A^0 - \beta A^1)\,,\quad A'^1 = \gamma\,(A^1 - \beta A^0)\,,\quad A'^2 = A^2\,,\quad A'^3 + A^3\,, \tag{1.85}$$

from where we can easily calculate

$$\varphi' = \gamma\,(\varphi - c\,\beta\,A_x)\,,\quad A'_x = \gamma\left(A_x - \frac{\beta}{c}\,\varphi\right)\,,\quad A'_y = A_y\,,\quad A'_z = A_z\,. \tag{1.86}$$

We would obtain the inverse transformation by substitution $\beta \to -\beta$.

The transformation of fields $\boldsymbol{E}$, $\boldsymbol{B}$ should be based on the transformation law for the tensor of the electromagnetic field $F^{\mu\nu}$

$$F'^{\mu\nu}(x') = \alpha^\mu_{.\rho}\,\alpha^\nu_{.\sigma}\,F^{\rho\sigma}(x). \tag{1.87}$$

By substituting into $\alpha^\mu{}_\nu$ we get transformational relations for its six independent components

$$F'^{01} = F^{01}\,,\quad F'^{02} = \gamma\,(F^{02} - \beta F^{12}),\quad F'^{03} = \gamma\,(F^{03} - \beta F^{13}),$$

$$F'^{12} = \gamma\,(F^{12} - \beta F^{02}),\quad F'^{13} = \gamma\,(F^{13} - \beta F^{03}),\quad F'^{23} = F^{23}.$$

If we substitute into $F^{\mu\nu}$ components of the fields according to the relation (1.64), we obtain

$$E'_x = E_x\,,\quad E'_y = \gamma\,(E_y - \beta\,c\,B_z)\,,\quad E'_z = \gamma\,(E_z + \beta\,c\,B_y)\,,$$

$$B'_x = B_x , \quad B'_y = \gamma \left(B_y + \frac{\beta}{c} E_z \right) , \quad B'_z = \gamma \left(B_z - \frac{\beta}{c} E_y \right) . \qquad (1.88)$$

In the case of general direction of velocity $\boldsymbol{V}$, we obtain different transformation relations for components parallel to $\boldsymbol{V}$ and perpendicular to $\boldsymbol{V}$:

$$\boldsymbol{E}'_{\parallel} = \boldsymbol{E}_{\parallel} , \quad \boldsymbol{E}'_{\perp} = \gamma \left(\boldsymbol{E}_{\perp} + \boldsymbol{V} \times \boldsymbol{B} \right) , \quad \boldsymbol{B}'_{\parallel} = \boldsymbol{B}_{\parallel} , \quad \boldsymbol{B}'_{\perp} = \gamma \left(\boldsymbol{B}_{\perp} - \frac{\boldsymbol{V}}{c^2} \times \boldsymbol{E} \right) . \qquad (1.89)$$

We obtain an inverse transformation by substituting $\boldsymbol{V} \rightarrow -\boldsymbol{V}$.

If we have a *pure electric field* (B = 0) in the system S, then in the system S' in addition to the electric field $\boldsymbol{E}'$, there is also the magnetic field $\boldsymbol{B}'$:

$$\boldsymbol{E}'_{\parallel} = \boldsymbol{E}_{\parallel} , \quad \boldsymbol{E}'_{\perp} = \gamma \, \boldsymbol{E}_{\perp} , \quad \boldsymbol{B}'_{\parallel} = 0, \quad \boldsymbol{B}'_{\perp} = - \gamma \frac{\boldsymbol{V}}{c^2} \times \boldsymbol{E} .$$

If $\boldsymbol{V} \perp \boldsymbol{E}$ is selected, then the generated field $\boldsymbol{B}'$ is perpendicular to $\boldsymbol{E}'$ and $\boldsymbol{V}$,

$$\boldsymbol{B}' = - \frac{\boldsymbol{V}}{c^2} \times \boldsymbol{E}' . \qquad (1.90)$$

Analogously, in the case of a *purely magnetic field*, $\boldsymbol{E} = 0$, $\boldsymbol{V} \perp \boldsymbol{B}$, an electric field is generated

$$\boldsymbol{E}' = \boldsymbol{V} \times \boldsymbol{B}' . \qquad (1.91)$$

In each transformation of quantities, we are concerned with the combinations of those quantities which do not change during the transformation and remain *relativistically invariant.* Thus, in the orthogonal transformation of coordinates in the Euclidean space, the distances of two points, the length are invariant, while in the Lorentz transformation in space-time, the interval is invariant. Let us try to find out what relativistic invariants can be created from the components of the electromagnetic field E_x, E_y, E_z, B_x, B_y, B_z. These components form the electromagnetic field tensor $F^{\mu\nu}$, or the dual tensor $F^{*\mu\nu}$, whose transformation properties are known, i.e. they are antisymmetrical contravariant four-tensors of the second order. We will obtain the simplest invariants that can be constructed by contraction. Since the diagonal elements of the antisymmetric tensor $F^{\mu\nu}$ are equal to zero,

$$I_0 = F^{\mu}_{\cdot\,\mu} = g_{\mu\nu} \, F^{\mu\nu} = 0.$$

Then

$$I_1 = F_{\mu\nu} \, F^{\mu\nu} = - F^*_{\mu\nu} \, F^{*\mu\nu} = 2 \left(\boldsymbol{B}^2 - \frac{\boldsymbol{E}^2}{c^2} \right), \qquad (1.92)$$

$$I_2 = F_{\mu\nu} \, F^{*\mu\nu} = \frac{4}{c} \, \boldsymbol{E} \cdot \boldsymbol{B}. \qquad (1.93)$$

We could thus construct other invariants by contraction the products of three, four and more $F^{\mu\nu}$ tensors, or the dual ones. However, these invariants would be either equal to zero or a combination of invariants I_1 and I_2. This fact is generally valid. The fact that the Lorentz group is locally isomorphic to the group $SL(2, C)$ of complex matrices 2 × 2 with determinant equal to one implies that the invariants of the tensor $F^{\mu\nu}$ are invariants of a complex matrix

$$\begin{pmatrix} F_z , & F_x - \mathrm{i}\, F_y \\ F_x + \mathrm{i}\, F_y , & -F_z \end{pmatrix} , \quad \text{kde} \ \boldsymbol{F} = \frac{\boldsymbol{E}}{c} + \mathrm{i}\, \boldsymbol{B},$$

with respect to complex linear transformations of the group $SL(2, C)$. According to the algebraic theory of invariants, matrix invariants are the coefficients of the characteristic equation

$$\begin{vmatrix} F_z - \lambda, & F_x - \mathrm{i}\, F_y \\ F_x + \mathrm{i}F_y, & F_z - \lambda \end{vmatrix} = \lambda^2 - \boldsymbol{F}^2 = 0$$

and the only (complex) coefficient of this characteristic equation

$$\boldsymbol{F}^2 = \frac{\boldsymbol{E}^2}{c^2} - \boldsymbol{B}^2 + \frac{2\mathrm{i}}{c} \boldsymbol{E} \cdot \boldsymbol{B} = \frac{-I_1 + \mathrm{i}I_2}{2},$$

determines *the two independent invariants* I_1 *and* I_2.

The knowledge of these invariants allows to perform a relativistically invariant classification of electromagnetic fields. These invariants can take on zero, positive or negative values, whose sign does not change in the Lorentz transformations. Therefore, one can distinguish nine cases (canonical forms) of the mutual position of vectors $\boldsymbol{E}$ and $\boldsymbol{B}$. If $I_1 = I_2 = 0$, then $E = cB$, $\boldsymbol{E} \cdot \boldsymbol{B} = 0$ in all Inertial systems. Both vectors are perpendicular to each other and correspond to the case of plane electromagnetic wave propagating in vacuum at a velocity c. The electromagnetic plane wave thus remains an electromagnetic plane wave in all inertial systems.

1.8 Actions for charged particle system and electromagnetic field

The actual solution to the problem of motion of charged particles that mutually interact through the electromagnetic field is very complex. However, the problem itself can easily be formulated in a compact form of Hamilton's variational principle. The apparatus of the Lagrangian formalism then allows to introduce (through the Noether theorem) basic preserving quantities. We will construct the respective action based on an analysis of two marginal cases:

A. A system of charged particles in a given external electromagnetic field.

B. An electromagnetic field generated by a system of charged particles moving in the prescribed manner.

In the case A, we know the Lagrange function for one particle

$$L \;=\; -\, m_0\, c^2\sqrt{1-\frac{v^2}{c^2}} \;-\; e(\varphi \;-\; \boldsymbol{v}\cdot\boldsymbol{A})\ ,$$

which we can easily generalize to N independent particles with rest masses m_α and charges e_α, $\alpha = 1,..., N$:

$$L_A \;=\; \sum_{\alpha=1}^{N}\left\{-\, m_\alpha\, c^2\,\sqrt{1-\frac{v_\alpha^2}{c^2}} \;-\; e_\alpha\,\left[\varphi(\boldsymbol{r}_\alpha,t) \;-\; \boldsymbol{v}_\alpha\cdot\boldsymbol{A}(\boldsymbol{r}_\alpha,t)\right]\right\}\ . \tag{1.94}$$

The mutual interactions of electromagnetic particles are not considered here, as they will be included in the resulting action through the field. In the case A, the charges e_α must be small enough not to affect the external field.

With the Lagrange function (1.94), the trajectory of particles will be determined by the Hamilton principle

$$\delta S_A \;=\; 0 \tag{1.95}$$

(for fixed-end variations) where the action integral

$$S_A \;=\; \int_{t_1}^{t_2} L_A\ \mathrm{d}t \tag{1.96}$$

is written as the sum of two terms

$$S_A \;=\; S_m \;+\; S_{mf}, \tag{1.97}$$

where S_m corresponds to particles and S_{mf} corresponds to interaction of particles with the external field (m – matter, f – field). Both parts are relativistically invariant:

$$S_m \;=\; \int_{t_1}^{t_2}\sum_\alpha(-\, m_\alpha\, c^2)\,\sqrt{1-\frac{v_\alpha^2}{c^2}}\ \mathrm{d}t \;=\; -\sum_\alpha m_\alpha\, c\int_{c_\alpha}\mathrm{d}s_\alpha,\quad \mathrm{d}s_\alpha \;=\; \sqrt{\mathrm{d}x_\alpha^\mu \mathrm{d}x_{\alpha\mu}}\ , \tag{1.98}$$

$$\begin{aligned} S_{mf} \;&=\; -\int_{t_1}^{t_2}\sum_\alpha e_\alpha\,\left[\varphi(\boldsymbol{r}_\alpha,t) \;-\; \boldsymbol{A}(\boldsymbol{r}_\alpha,t)\cdot\dot{\boldsymbol{r}}_\alpha\right]\,\mathrm{d}t \;= \\ &=\; -\sum_\alpha e_\alpha\int_{t_1}^{t_2} A_\mu(\boldsymbol{r}_\alpha,t)\,\frac{\mathrm{d}x_\alpha^\mu}{\mathrm{d}t}\mathrm{d}t \;=\; -\sum_\alpha e_\alpha\int_{c_\alpha} A_\mu(x_\alpha)\ \mathrm{d}x_\alpha^\mu\ . \end{aligned} \tag{1.99}$$

We see that interaction occurs only where charges are present. In the resulting expressions, the integration is performed through the world lines c_a of individual particles between the hyperplanes $x^0 = ct_1$, $x^0 = ct_2$.

Using the expression for charge density of point particles with trajectories $\boldsymbol{r}_\alpha = \boldsymbol{r}_\alpha(t)$

$$\rho(r,\ t) \ = \ \sum_\alpha e_\alpha \ \delta^3(r - r_\alpha(t))$$

we will transform S_{mf} into

$$S_{mf} \ = \ - \int\limits_{R^3} \left[\int\limits_{t_1}^{t_2} \rho A_\mu(x) \ \frac{\mathrm{d}x^\mu}{\mathrm{d}t} \ \mathrm{d}t \right] \mathrm{d}V \ , \tag{1.100}$$

and using the definition of the four-current (1.55), we will transform it to the form

$$S_{mf} \ = \ - \ \frac{1}{c} \int\limits_{V*} j^\mu \ A_\mu \ \mathrm{d}V^* \ = \ - \int\limits_{t_1}^{t_2} \int\limits_{R^3} (\rho\varphi - \rho \boldsymbol{v} \cdot \boldsymbol{A}) \mathrm{d}V \ \mathrm{d}t, \tag{1.101}$$

which will then be used in situation B including the case of continuous distribution of charges and currents. Let us remind that action for the relativistic field is $S = (1/c) \int \mathcal{L} \, \mathrm{d}V^*$, where $\mathrm{d}V^* = c \, \mathrm{d}t \, \mathrm{d}V$. Thus, the interaction part of the action corresponds to the density of the Lagrange function

$$\mathcal{L}_{mf} \ = \ - \ j^\mu \ A_\mu \tag{1.102}$$

referred to as *an interaction Lagrangian.*

Let us demonstrate that in the case of a single particle, variation of the action S_A with respect to x^μ yields the equation of motion (1.69). Let us write the action S_A in the form

$$S_A \ = \ \int\limits_{t_1}^{t_2} \left[-m_0 \ c \ \sqrt{\frac{\mathrm{d}x^\mu}{\mathrm{d}t} \ \frac{\mathrm{d}x_\mu}{\mathrm{d}t}} \ - \ e \ A_\mu(x) \ \frac{\mathrm{d}x^\mu}{\mathrm{d}t} \right] \ \mathrm{d}t$$

and then we will proceed to the integral through the proper time of the particle τ

$$S_A = \int\limits_{\tau_1}^{\tau_2} (-m_0 \ c \ \sqrt{u^\mu u_\mu} \ - \ e \ A_\mu(x) \ u^\mu) \ \mathrm{d}\tau \ ,$$

in whose integrand there is the Lagrange function

$$L \ = \ L(x^\mu, u^\mu, \tau) \ = \ T \ - \ U^* \ = \ - \ m_0 \ c \ \sqrt{u^\mu u_\mu} \ - \ e \ A_\mu(x) \ u^\mu \ , \qquad u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$$

with U* velocity-dependent potential. The action principle (1.95) is equivalent to the Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \left(\frac{\partial L}{\partial u_\nu} \right) \ - \ \frac{\partial L}{\partial x_\nu} \ = \ 0, \qquad \text{tj.} \quad \frac{\mathrm{d}}{\mathrm{d}\tau} \left(\frac{\partial T}{\partial u_\nu} \right) \ = \ - \frac{\partial U^*}{\partial x_\nu} \ + \ \frac{\mathrm{d}}{\mathrm{d}\tau} \ \frac{\partial U^*}{\partial u_\nu} \ .$$

The transformation of both sides

$$- \ m_0 \ c \ \frac{\mathrm{d}}{\mathrm{d}\tau} \left(\frac{\partial}{\partial u_\nu} \sqrt{u^\mu u_\mu} \right) \ = \ - \ m_0 \ c \ \frac{\mathrm{d}}{\mathrm{d}\tau} \left(\frac{u^\nu}{\sqrt{u^\mu u_\mu}} \right) \ = \ - \ m_0 \ \frac{\mathrm{d}u^\nu}{\mathrm{d}\tau} \ , \qquad u^\mu u_\mu \ = \ c^2 \ ,$$

$$- \ e \ \frac{\partial A^\mu}{\partial x_\nu} \ u_\mu \ + \ e \ \frac{\mathrm{d}A^\nu}{\mathrm{d}\tau} \ = \ - \ e \ \left(\frac{\partial A^\mu}{\partial x_\nu} \ - \ \frac{\partial A^\nu}{\partial x_\mu} \right) \ u_\mu \ = \ - \ e \ F^{\nu\mu} \ u_\mu \ = \ - \ K^\nu$$

actually leads to the covariant motion equation (1.69) with the Lorentz four-force.

How the action in the case B should be constructed? We will look for an action principle which would serve as a source for Lorentz-Maxwell's equations for the electromagnetic field, whose source would be the specified four-current j^μ. In mechanics, the Lagrange function is the scalar function of general coordinates and velocities determining the state of the system v at any time t. In electrodynamics, the state of the field in vacuum at time t is determined either by the values of the fields $\boldsymbol{E}$, $\boldsymbol{B}$, or by potentials A^μ and their first derivatives $A_{\mu,\nu} = \partial A_\mu / \partial x^\nu$ at any point in space. In order to make the Lagrange equations *linear*, the sought density of the Lagrange function $\mathcal{L}_B$ can be at most quadratic with respect to the field variables.

An important limitation is the *covariance* of the resulting equations: to do this, the scalar $\mathcal{L}_B$ must be an invariant. As we already know, only two independent invariants can be constructed from the electromagnetic field tensor $F^{\mu\nu}$:

$$F_{\mu\nu} F^{\mu\nu} \ = \ 2 \ \left(B^2 \ - \ \frac{E^2}{c^2} \right), \qquad F^*_{\mu\nu} F^{\mu\nu} \ = \ \frac{4}{c} \ \boldsymbol{E} \cdot \boldsymbol{B}. \tag{1.103}$$

Both of them are quadratic, but the second is a pseudoscalar, and therefore it is not applicable. Quadratic invariant $A_\mu A^\mu$ also cannot be used because it violates the gauge invariance.

As a candidate for the field part of the density $\mathcal{L}_f$ of the Lagrange function

$$\mathcal{L}_B = \mathcal{L}_f + \mathcal{L}_{mf} \tag{1.104}$$

there remains

$$\mathcal{L}_f = \alpha F_{\mu\nu} F^{\mu\nu}. \tag{1.105}$$

The constant a is determined so that the Lagrange equations for $\mathcal{L}_B$ relative to the general coordinates A^μ,

$$\frac{\partial \mathcal{L}_B}{\partial A_\mu} - \frac{\partial}{\partial x^\nu} \left(\frac{\partial \mathcal{L}_B}{\partial A_{\mu,\nu}} \right) = 0$$

led precisely to the equation of the first series (1.66). Since

$$\frac{\partial \mathcal{L}_B}{\partial A_\mu} = - j^\mu \ , \qquad \frac{\partial \mathcal{L}_B}{\partial A_{\mu,\nu}} = - 4\ \alpha\ F^{\mu\nu}$$

we obtain $\alpha = -1/(4\mu_0)$, and therefore the Lagrangian of the electromagnetic field equals

$$\mathcal{L}_f = - \frac{1}{4\mu_0} F_{\mu\nu}\ F^{\mu\nu} = \frac{1}{2} \left(\varepsilon_0\ E^2 - \mu_0\ H^2 \right) . \tag{1.106}$$

We see that in the case A we had an action (1.97), while in case B we have an action

$$S_B = S_f + S_{mf} \ , \tag{1.107}$$

where

$$S_f = - \frac{1}{4\ \mu_0\ c} \int_{V^*} F_{\mu\nu}\ F^{\mu\nu}\ \mathrm{d}V^* \tag{1.108}$$

and S_{mf} was introduced by the relation (1.101).

In a general situation, simultaneous equations of motion of both types A and B can be obtained by variation of the action

$$S = S_f + S_m + S_{mf} \ . \tag{1.109}$$

As we have seen, the variation S according to field variables - potentials of the electromagnetic field (with fixed mechanical variables) yields the Lorentz-Maxwell equation (1.66), whereas the variation according to according to particle variables (with fixed field variables) yields the mechanical equations of motion (1.69). We performed both variations in this paragraph by constructing the corresponding Lagrange equations.

We can compose (6.137) a *canonical (antisymmetric) energy-momentum tensor* from the derived action S_f for the electromagnetic field (free, non-source)

$$T_\mu^{\cdot\nu} = \frac{\partial \mathcal{L}_f}{\partial A_{\rho,\nu}} A_{\rho,\mu} - \delta_\mu^\nu\ \mathcal{L}_f = \frac{1}{\mu_0} A_{\rho,\mu}\ F^{\rho\nu} + \frac{1}{4\ \mu_0} F_{\rho\sigma}\ F^{\rho\sigma}\ \delta_\mu^\nu .$$

We see that it is not gauge-invariant, but by adding the four-divergence

$$- \frac{1}{\mu_0} A_{\mu,\rho}\ F^{\rho,\nu} = \frac{1}{\mu_0} \frac{\partial}{\partial x^\rho} (A_\mu\ F^{\nu\rho}) \equiv \frac{\partial Q_\mu^{\cdot\nu\rho}}{\partial x^\rho}$$

we create a gauge-invariant *symmetrical tensor of energy-momentum* according to (6.152)

$$T^{\mu\nu} = \frac{1}{\mu_0} \left(- g_{\rho\sigma}\ F^{\mu\rho}\ F^{\nu\sigma} + \frac{1}{4}\ g^{\mu\nu}\ F_{\rho\sigma}\ F_{\rho\sigma} \right) . \tag{1.110}$$

By substituting for $F^{\mu\nu}$ we find that the tensor $T^{\mu\nu}$ has a known structure (6.159)

$$(T^{\mu\nu}) \; = \; \left(\begin{array}{ccc} w & \vdots & \frac{\boldsymbol{S}}{c} \\ \cdots & \cdots & \cdots \\ \frac{\boldsymbol{S}}{c} & \vdots & (\sigma_{ik}) \end{array} \right) \; , \tag{1.111}$$

where w, S and (σ_{ik}) are the quantities derived in paragraph 7.4. for the electromagnetic field in vacuum. The tensor $T^{\mu\nu}$ has zero four-divergence in a free field (6.157), but with the sources present we can derive the equation

$$\frac{\partial T^{\mu\nu}}{\partial x^{\nu}} \; = \; - \, f^{\mu} \; , \tag{1.112}$$

where f^{μ} is covariant Lorentz force density (1.68). This important covariant equation summarizes, in a nutshell, the *local laws of energy and momentum conservation* for a system of charged particles and the electromagnetic field in vacuum, which we derived in paragraph 7.4.

1.9 Problems

1 *Motion of a charged non-relativistic particle in electric and magnetic field.* Solve equations of motion for a non-relativistic charged particle with mass m and charge e moving a) in a constant homogeneous electric field $\boldsymbol{E} = (E, 0, 0)$ (initial conditions $\boldsymbol{r} = 0$, $\dot{\boldsymbol{r}} = \boldsymbol{v}_0$ for $t = 0$); b) in a constant homogeneous magnetic field $\boldsymbol{B} = (0, 0, B)$ (initial conditions $\boldsymbol{r} = 0$, $\dot{\boldsymbol{r}} = (\boldsymbol{v}_0, 0, 0)$).
[a) $x = \frac{eE}{2m} t^2 + v_{0x}t$, $y = v_{0y}t$, $z = v_{0z}t$; b) $x = \frac{v_0}{\alpha} \sin\alpha t$, $y = -\frac{v_0}{\alpha}(1 - \cos\alpha t)$, $z = 0$; $\alpha = \frac{eB}{m}$]

2 *Žáček's magnetron.* Demonstrate that the trajectory of a particle of mass m and charge e in crossed constant homogeneous fields $\boldsymbol{E} = (0, E, 0)$, $\boldsymbol{B} = (0, 0, B)$ is a cycloid, if initially $\boldsymbol{r}(0) = (0,0, z_0)$, $\dot{\boldsymbol{r}}(0) = 0$. This cycloid is generated by rolling a circle with radius $r_0 = mE/eB^2$ on surface $z = z_0$ along the x-axis with the angular frequency $\omega_c = eB/m$ (the cyclotron frequency).
[$x = \frac{E}{\omega_c B}(\omega_c t - \sin\omega_c t)$, $y = \frac{E}{\omega_c B}(1 - \cos\omega_c t)$, $z = z_0$]

3 *Normal Zeeman effect.* Electron of mass $m \doteq 9{,}1.10^{-31}$ kg and charge $e \doteq -1{,}6.10^{-19}$ C bound to the origin by force $-k\boldsymbol{r}$ is harmoniously oscillating with angular frequency $\omega_0 = \sqrt{k/m}$ (isotropic harmonic oscillator). Determine how the angular frequency of oscillations changes if this spatial oscillator is placed in a constant homogeneous magnetic field $\boldsymbol{B} = (0,0, B)$. *Instructions:* Write the equations of motion of the electron. Look for solutions in the x_1x_2 plane in the form $x_i = A_i \exp(i\omega t)$, $i = 1, 2$. To determine ω, use the condition $|eB/2m| \ll \omega_0$. Consider to what extent this condition is met for an electron emitting visible light and placed in a magnetic field ~ 1T.
[In the plane x_1x_2: $\omega \doteq \omega_0 \pm \frac{eB}{2m}$, ve směru osy x_3: $\omega = \omega_0$]

4 *Constant homogeneous magnetic field.* Show that the vector potential $\boldsymbol{A} = \boldsymbol{B} \times \boldsymbol{r} / 2$ determines a constant homogeneous magnetic field $\boldsymbol{B}$. When selecting a coordinate system such that $\boldsymbol{B} = (0,0, \boldsymbol{B})$, where $\boldsymbol{B}$ = const, specify: a) Cartesian components A_x, A_y, A_z, b) components A_R, A_φ, A_z in cylindrical coordinates R, φ, z. *Instructions:* Components A_R, A_φ, A_z are given by orthogonal projections of $\boldsymbol{A}$ into the unit vectors $\boldsymbol{e}_R$, $\boldsymbol{e}_\varphi$, $\boldsymbol{e}_z$ in the directions of the coordinate curves, i.e. in the directions $\partial\boldsymbol{r}/\partial R$, $\partial\boldsymbol{r}/\partial\phi$, $\partial\boldsymbol{r}/\partial z$.
[a) $\boldsymbol{A} = (-By/2, Bx/2, 0)$, b) $A_R = A_z = 0$, $A_\varphi = BR/2$]

5 *Constant homogeneous magnetic field.* Characterize the class of vector potentials $\boldsymbol{A}$, which determine the constant homogeneous magnetic field $\boldsymbol{B} = (0,0, B)$, B = const. Do they include $\boldsymbol{A}' = (-By, 0,0)$?
[$A' = (-By/2, Bx/2, 0) + \mathrm{grad}\Lambda(x, y, z)$, $\Lambda = Bxy/2$]

6 *Preserving quantities in a constant magnetic field.* The vector potential $\boldsymbol{A} = \boldsymbol{B} \times \boldsymbol{r} / 2$ determines a constant homogeneous magnetic field $\boldsymbol{B}$. Write a Hamiltonian $H(\boldsymbol{r}, \boldsymbol{p})$ of a particle a mass of mass m with charge q in this magnetic field. Prove that vector quantity $\boldsymbol{p} + q\boldsymbol{A}$ represents three integrals of motion of this system. *Instructions:* Select the axis z in the direction of the magnetic field.

7 Show that when a relativistic particle with charge q and rest mass m_0 is moving in an external magnetic field determined by vector potential $\boldsymbol{A} = (0,0, A(x, y))$, the quantity $p_z = \frac{m_0 v_z}{\sqrt{1-\frac{v^2}{c^2}}} + qA(x, y)$. is preserved.

Instructions: Determine the canonical momentum corresponding to the cyclic coordinate z.

8 *Magnetic flux through the surface.* Use the Stokes theorem to specify the physical meaning of the line $\oint_\Gamma \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l}$, d$\boldsymbol{l}$, where $\boldsymbol{A}$ is the vector potential of a magnetic field and Γ is a closed curve bounding a two-dimensional surface f. What physical quantity does this integral determine? Is this physical quantity gauge-invariant?
[$\oint_\Gamma \boldsymbol{A}.\mathrm{d}\boldsymbol{l} = \int_f \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{f}$]

9 *Singular magnetic field.* Let the vector potential in the area of space R^3, where $x \neq 0$, $y \neq 0$, have components

$$A_x = -\frac{\Phi}{2\pi}\frac{y}{x^2 + y^2}, \qquad A_y = \frac{\Phi}{2\pi}\frac{x}{x^2 + y^2}, \qquad A_z = 0.$$

a) Determine the magnetic field in this area. b) Calculate the line integral $\oint_\Gamma \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l}$, of this vector potential along the circle $x^2 + y^2 = r^2 \neq 0$. Explain which singular magnetic field this vector potential corresponds to. *Instructions:* Integrate in polar coordinates.

[a) $\boldsymbol{B} = \boldsymbol{0}$, b) Φ]

10 *Coulomb potential in anisotropic medium*. Determine the electrostatic potential field induced by a point charge e located at the origin in a homogeneous anisotropic dielectric medium with relative permittivity tensor ε_{ik}, $D_i = \varepsilon_0 \sum_k \varepsilon_{ik} E_k$. *Instructions:* Derive the Poisson equation in the principal axes of the tensor ε_{ik}. By substituting $x_i = x_i' \sqrt{\varepsilon_i}$ transform into the usual form of the Poisson equation.

$$\left[\ \Delta'\varphi(\boldsymbol{r}') = -\frac{e}{\varepsilon_0\sqrt{\varepsilon_1\varepsilon_2\varepsilon_3}}\delta^3(\boldsymbol{r}') \qquad \Rightarrow \qquad \varphi(\boldsymbol{r}) = \frac{e}{\varepsilon_0\sqrt{\varepsilon_1\varepsilon_2\varepsilon_3}}\frac{1}{\sqrt{\frac{x_1^2}{\varepsilon_1}+\frac{x_2^2}{\varepsilon_2}+\frac{x_3^2}{\varepsilon_3}}}\right.$$

and in the general Cartesian coordinates with the matrix $\varepsilon = (\varepsilon_{ik})$ we can write

$$\left.\varphi(\boldsymbol{r}) = \frac{e}{4\pi\varepsilon_0\sqrt{|\varepsilon|\sum_{ik}(\varepsilon^{-1})_{ik}x_i x_k}}.\ \right]$$

11 *The electric dipole moment.* Electric dipole moment of the charges distributed in the bounded volume V with the density $\rho(\boldsymbol{r}, t)$ is $\rho(t) = \int_V \boldsymbol{r}\rho(\boldsymbol{r}, t)\mathrm{d}V$. a) Determine the electric dipole moment of the point charge system at points $r_\alpha(t)$, $\alpha = 1, \dots, n$. b) Prove that the electric dipole moment of a neutral charge system does not depend on the origin selection. *Instructions:* Translate the coordinate system by the formula $\boldsymbol{r}' = \boldsymbol{r} + \boldsymbol{a}$. c) Prove that the electric dipole moment of the point symmetric charge distribution $\rho(-\boldsymbol{r}) = \rho(\boldsymbol{r})$ is equal to zero.

[a) $\boldsymbol{p}(t) = \sum_\alpha e_\alpha \boldsymbol{r}_\alpha(t)$]

12 *Stationary current.* Show that in the case of stationary currents div$\boldsymbol{j} = 0$ in bounded volume V, $\int \boldsymbol{j}\mathrm{d}V = 0$ is valid. *Instructions:* Divide the current into closed current loops for which $\int \boldsymbol{j}\mathrm{d}V = \oint I\mathrm{d}l = 0$

13 *The magnetic dipole moment* of current distribution $\boldsymbol{j}(\boldsymbol{r}, t)$ in the finite volume V is defined by the relation:

$$\boldsymbol{m}(t) = \tfrac{1}{2}\int_V \boldsymbol{r}\times\boldsymbol{j}(\boldsymbol{r},t)\mathrm{d}V.$$

a) What is the magnetic dipole moment of the system N of point charges e_α located at points $\boldsymbol{r}_\alpha(t)$ and moving at velocities $\boldsymbol{v}_\alpha(t)$? b) What is the magnetic dipole moment of the closed plane curve Γ with linear current I flowing though it? *Instructions: $\boldsymbol{j}\ dV = I\ dl$.*

[a) $\boldsymbol{m}(t) = \frac{1}{2}\sum_{\alpha=1}^{N} e_\alpha \boldsymbol{r}_\alpha(t)\times\boldsymbol{v}_\alpha(t)$, b) $\boldsymbol{m} = \frac{I}{2}\oint_\Gamma \boldsymbol{r}\times\mathrm{d}\boldsymbol{l} = IS\boldsymbol{n}$,
where S is the surface bounded by a loop and n is its normal.]

14 *Pulling a dielectric medium between capacitor plates* Calculate how the energy of the electrostatic field will change, if we fill the capacitor space with a homogeneous soft dielectric medium. *Instructions:* compare the solution of Maxwell's electrostatic equations in vacuum and in a dielectric medium.

15 *Force in the capacitor* Plate capacitor consists of two parallel conductive plates with area S carrying charges $+Q$ and $-Q$. Plates are placed in a dielectric medium with permittivity ε. If the plates are large and their distance is small, then the electric field is concentrated practically between the plates and is homogeneous. Calculate the forces with which the plates interact. *Instructions:* Calculate the capacitor field energy and its change when the plates spacing changes: a) if the plates are isolated and their charge is constant, b) if the plates have a constant potential difference φ_0. Based on this, calculate the forces.

[a) $Q =$konst.: $(\mathrm{d}W)_a = Q^2\mathrm{d}l/(2\varepsilon S)$, b) $\varphi_0 =$konst.: $(\mathrm{d}W)_b = -\frac{1}{2}\varepsilon S\varphi_0^2\mathrm{d}l/l^2$;
$(\mathrm{d}W)_a = -(\mathrm{d}W)_b$, $F_a/S = -F_b/S = \varepsilon E^2/2$]

16 *Maxwell stress tensor* Analyze the meaning of Maxwell stress tensor of electrostatic field in a homogeneous soft dielectric medium. Using the relation (1.54), show that in the direction of field E, the dielectric medium is subjected to tensile stress $\varepsilon E^2/2$, while in the direction perpendicular to $\boldsymbol{E}$, it is subjected to pressure stress of the same magnitude. (Analogous results also apply to a constant magnetic field.)

$$\left[\ \boldsymbol{F} = \oint_{\partial V}[\varepsilon\boldsymbol{E}(\boldsymbol{E}\cdot\boldsymbol{n}) - \frac{1}{2}\varepsilon E^2\boldsymbol{n}]\mathrm{d}f;\right.$$

in cases E ∥ d***f*** and ***E*** ⊥ d***f***, where d***f*** = ***n***d*f*, we obtain

$$\boldsymbol{F}_{\parallel} = \oint\limits_{\partial V} \frac{1}{2}\varepsilon E^2 \boldsymbol{n}\mathrm{d}f = -\boldsymbol{F}_{\perp}. \Bigg]$$

7 *Electromagnetic radiation of the Sun* near the Earth has an intensity I ≐ 1365 W m^{-2} called solar irradiance (formerly a solar constant). Based on this, calculate the mean values of the magnitude of the electric field and the magnetic field in the electromagnetic field of solar radiation near the Earth. These are the quantities $E = \sqrt{\langle E^2 \rangle}$ I $B = \sqrt{\langle B^2 \rangle}$ ($\mu_0 = 4\pi \cdot 10^{-7}$ N A^{-2}).
[510 V/m, $1,7 \cdot 10^{-6}$ T]

18 *Potentials of a moving charge*. Calculate the potentials of the electromagnetic field induced in vacuum by a point particle with charge q, which moves uniformly in a straight line at the velocity $\boldsymbol{V} = (V, 0,0)$ in the inertial system S of an observer. What is the form of the equipotential surfaces of the scalar potential? *Instructions:* Use the rest system S' of a particle located at the origin of the system S' and the special Lorentz transformation $S' \to S$ of the four-potential $A^\mu = (\varphi/c, \boldsymbol{A})$.

$$\left[\varphi = \frac{q}{4\pi\varepsilon_0\sqrt{(x-Vt)^2 + (1-\beta^2)(y^2+z^2)}}, \quad \boldsymbol{A} = (\boldsymbol{V}/c^2)\varphi\right]$$

19 Field of a *moving charge* Calculate the vectors $\boldsymbol{E}(x, y, z, t)$, $\boldsymbol{B}(x, y, z, t)$ of the electromagnetic field in the inertial system S of the observer, if it is induced in vacuum by a point particle with charge q, which is moving uniformly in a straight line at the velocity $\boldsymbol{V} = (V, 0, 0)$ in the system S. *Instructions:* Use the formula for the Coulomb electric field **E'** induced in the inertial system S' by a particle located in the beginning of the system S', and a special Lorentz transformation $S' \to S$ of the electromagnetic field tensor components $F'^{\mu\nu}(x'^\lambda)$.

$$\left[\boldsymbol{E} = \frac{q(1-\beta^2)(\boldsymbol{r} - \boldsymbol{V}t)}{4\pi\varepsilon_0[(x-Vt)^2 + (1-\beta^2)(y^2+z^2)]^{3/2}}, \quad \boldsymbol{B} = \frac{\boldsymbol{V}}{c^2} \times \boldsymbol{E}\right]$$

Chapter 2

Electromagnetic waves

20 Plane electromagnetic waves

In order to understand the physical meaning of Maxwell's equations, we need to know their solutions under different physical conditions. Whenever we solve them, we will learn something new about their character. It is usually recommended to visualise the solution e.g. using lines of force. In this way we gradually get to a real physical understanding of the equations. If you analyze equations mathematically, you should not think you already understand physics. The real physical situations of the real world are so complex that equations need to be understood much deeper. Physical conditions also sometimes allow to replace the solution to the complex problem by an approximation, and our task is then to assess the conditions for the applicability of such approximate solutions.

In the stationary case, Maxwell's equations do not give anything new that could not be derived from the Coulomb's law for charges and the Ampère's force law. In the basic course of physics, the main types of electric and magnetic fields are derived, which correspond to the stationary configurations that most often occur in practice. This chapter is therefore devoted solely to to the time-varying fields, which bring new consequences typical for *non-stationary Maxwell's equations*: the existence of electromagnetic waves and their radiation with accelerated charged particles.

Let us begin by solving Maxwell's equations in an 'empty space', that is, in an area where there are no free sources

$$\rho = 0, \qquad \boldsymbol{j} = 0.$$

Maxwell's equations (in a homogeneous isotropic soft medium with material constants ε, μ) obviously have a trivial constant solution. However, we will demonstrate that there are non-zero time-dependent solutions. Using the method of potentials, we can substitute Maxwell's equations with a simple equivalent system in the Coulomb gauge, where $\boldsymbol{j} = 0$,

$$\begin{aligned} \varphi &= 0, \\ \Delta \boldsymbol{A} - \varepsilon\mu \frac{\partial^2 \boldsymbol{A}}{\partial t^2} &= 0, \\ \mathrm{div} \boldsymbol{A} &= 0. \end{aligned}$$

It is therefore necessary to seek for a solution of the *wave equation* (2.2) between vector potentials A fulfilling the Coulomb gauge condition (2.3). Fields $\boldsymbol{E}$, $\boldsymbol{B}$ are then calculated by substituting $\varphi = 0$ and the obtained solution $\boldsymbol{A}$ to the relations

$$\boldsymbol{E} = -\mathrm{grad}\varphi - \frac{\partial \boldsymbol{A}}{\partial t} = -\frac{\partial \boldsymbol{A}}{\partial t}, \qquad \boldsymbol{B} = \mathrm{rot}\boldsymbol{A}. \tag{2.4}$$

The basic solution of the wave equation is the well-known *d'Alembert's solution* which represents the *traveling plane wave.* For the wave equation (2.2)—or, actually, three wave equations for components A_1, A_2, A_3—such a solution is given by *any vector function* $\boldsymbol{A}(\xi)$ of the *scalar argument*

$$\xi = \boldsymbol{s} \cdot \boldsymbol{r} - vt, \qquad |\boldsymbol{s}| = 1, \tag{2.5}$$

called the *wave phase.* The unit vector with has a simple geometric meaning: it determines the direction of the normal of the *constant phase planes*

$$\xi = \boldsymbol{s} \cdot \boldsymbol{r} - vt = C, \tag{2.6}$$

on which the vector function $\boldsymbol{A}(\xi)$ is constant. The real parameter in has a velocity dimension and is called a *phase velocity*, since it indicates the velocity of propagation of the planes of a constant phase. Indeed, let us observe the plane (2.6) in two moments of time $t_1 < t_2$, Fig.8.1. Both planes $\boldsymbol{s} \cdot \boldsymbol{r}_1 - vt_1 = C$, $\boldsymbol{s} \cdot \boldsymbol{r}_2 - vt_2 = C$ are parallel according to (2.6) (both have a normal $\boldsymbol{s}$) and their distance l increases at velocity v,

$$l = \boldsymbol{s} \cdot (\boldsymbol{r_2} - \boldsymbol{r_1}) = v(t_2 - t_1).$$

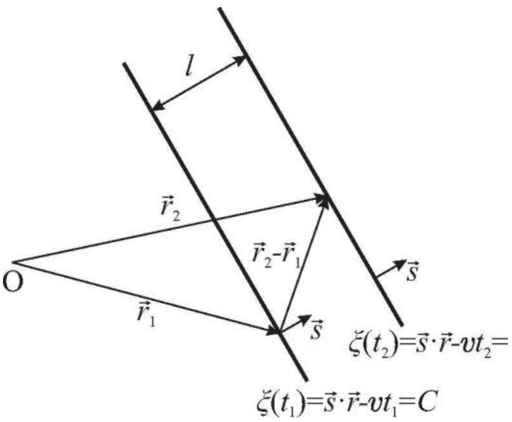


Figure 2.1: Phase velocity

The dependence of phase velocity in the material constants ε, μ can be found by substituting $\boldsymbol{A}(\xi)$ into the wave equation (2.2). Since

$$\Delta \boldsymbol{A} = \sum_{k=1}^{3} \frac{\partial^2 \boldsymbol{A}(\xi)}{\partial x_k^2} = \sum_{k=1}^{3} s_k^2 \frac{\mathrm{d}^2 \boldsymbol{A}(\xi)}{\mathrm{d}\xi^2} = \frac{\mathrm{d}^2 \boldsymbol{A}(\xi)}{\mathrm{d}\xi^2}, \qquad \frac{\partial^2 \boldsymbol{A}(\xi)}{\partial t^2} = v^2 \frac{\mathrm{d}^2 \boldsymbol{A}(\xi)}{\mathrm{d}\xi^2},$$

we have an identity

$$\Delta \boldsymbol{A} - \varepsilon\mu \frac{\partial^2 \boldsymbol{A}}{\partial t^2} = (1 - \varepsilon\mu v^2) \frac{\mathrm{d}^2 \boldsymbol{A}(\xi)}{\mathrm{d}\xi^2} = 0.$$

Since generally $\mathrm{d}^2\boldsymbol{A}(\xi)/\mathrm{d}\xi^2 \neq 0$, we obtain a *Maxwell's relation*

$$v^2 = \frac{1}{\varepsilon\mu}. \tag{2.7}$$

In vacuum, the phase velocity of an electromagnetic wave is numerically equal to the speed of light in vacuum, since

$$v^2 = \frac{1}{\varepsilon_0\mu_0} = c^2, \quad c \doteq 3 \cdot 10^8 \text{ m s}^{-1}.$$

On this basis, Maxwell pronounced the hypothesis that light is an electromagnetic field, thereby establishing an *electromagnetic theory of light.* The propagation of electromagnetic waves in material medium is usually characterized by a *refractive index n* defined by the relation

$$v = \frac{c}{n}, \quad \text{tj.} \qquad n = \frac{c}{v} = \sqrt{\varepsilon_r\mu_r}. \tag{2.8}$$

Since for transparent substances $\mu_r \doteq 1$, we have $n \doteq \sqrt{\varepsilon_r}$.

The vector potential $\boldsymbol{A}(\xi)$ solves the equation (2.2) but still has to fulfill the gauge condition (2.3),

$$\mathrm{div}\boldsymbol{A}(\xi) = \sum_{k=1}^{3} s_k \frac{\mathrm{d}A_k(\xi)}{\mathrm{d}\xi} = \boldsymbol{s} \cdot \frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}\xi} = 0. \tag{2.9}$$

Thus, we see that the vector function d $\boldsymbol{A}(\xi)/\mathrm{d}\xi$ is still perpendicular to the direction of propagation $\boldsymbol{s}$. Now, according to the equation (2.4), we will calculate vectors $\boldsymbol{E}$ and $\boldsymbol{B}$:

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}(\xi)}{\partial t} = v\frac{\mathrm{d}\boldsymbol{A}(\xi)}{\mathrm{d}\xi}, \quad \boldsymbol{B} = \mathrm{rot}\boldsymbol{A}(\xi) = \boldsymbol{s} \times \frac{\mathrm{d}\boldsymbol{A}(\xi)}{\mathrm{d}\xi}. \tag{2.10}$$

The expressions (2.9) and (2.10) indicate *that vectors **s**, **E**, **B** respectively, form a right-handed system of mutually orthogonal vectors.* Here vector relations apply

$$\boldsymbol{B} = \frac{1}{v}\boldsymbol{s} \times \boldsymbol{E} \quad \text{resp.} \quad \boldsymbol{E} = v\boldsymbol{B} \times \boldsymbol{s},$$

therefore, the vector magnitudes are connected by a relation

$$E = vB \quad \text{resp.} \quad \sqrt{\varepsilon}E = \sqrt{\mu}H. \tag{2.11}$$

The ratio of the intensity vectors magnitudes ***E*** and ***H*** defines *the characteristic impedance Z* of the medium. According to the

$$Z = \frac{E}{H} = \sqrt{\frac{\mu}{\varepsilon}}. \tag{2.12}$$

The characteristic impedance of vacuum is numerically equal

$$Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \doteq 377\ \Omega;$$

note that $Z_0 \doteq 120\pi\ \Omega$. When describing the propagation of electromagnetic waves, it is often expedient to replace the material constants ε, μ with wave constants $v = 1/\sqrt{\varepsilon\mu}$, $Z = \sqrt{\mu/\varepsilon}$.

> *The electromagnetic field **E**, **B** of a traveling plane wave (**E** ⊥ **B**, E = vB) is oriented transversely to the direction of propagation **s**, it is constant along the planes of constant phase, and this signal propagates in the direction of the finite velocity* $v = 1/\sqrt{\varepsilon\mu}$.

It is important to know the relevant *energy quantities* in order to understand the physical effects of electromagnetic waves. With respect to equations (2.11) and (2.12), for the energy density w and the energy flow density ***S*** in the plane wave, we obtain

$$w = \frac{1}{2}(\varepsilon \boldsymbol{E}^2 + \mu \boldsymbol{H}^2) = \varepsilon \boldsymbol{E}^2, \tag{2.14}$$

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H} = \frac{1}{Z}\boldsymbol{E} \times (\boldsymbol{s} \times \boldsymbol{E}) = \frac{1}{Z}\boldsymbol{E}^2\boldsymbol{s}. \tag{2.15}$$

Thus, the relation between ***S*** and w in the plane wave

$$\boldsymbol{S} = \frac{1}{Z\varepsilon}w\boldsymbol{s} = wv\boldsymbol{s} \tag{2.16}$$

is valid. It expresses the fact that the energy of the field of a plane wave flows in the direction ***s*** with velocity v. According to Fig 8.2 it is evident, that the energy $w\mathrm{d}V$ contained in the cylinder with height $v\mathrm{d}t$ and volume $\mathrm{d}V = v\mathrm{d}t \cdot 1$ passes perpendicularly to unit area $\xi(t) = C$ at the velocity v per time dt. This energy per time unit is equal to $w\mathrm{d}V/\mathrm{d}t = wv$ and represents the density of the energy flux.

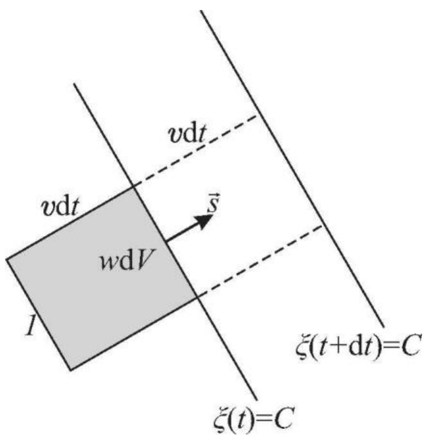


Figure 2.2: Energy flux density

Radiation pressure

According to chap. 7 and Equation (2.16), the momentum density of the electromagnetic field of the plane wave in vacuum is

$$\boldsymbol{g} = \frac{\boldsymbol{S}}{c^2} = \frac{1}{c^2} wc\ \boldsymbol{s}\ =\ \frac{w}{c}\ \boldsymbol{s}. \tag{2.17}$$

Thus, the electromagnetic plane wave transmits momentum along with energy. It can be therefore expected to exert pressure at incidence on material bodies. Let us suppose that, as shown in figure 8.3, a massive body has a perfectly absorbing plane surface, where the electromagnetic field is absorbed at a shallow depth.

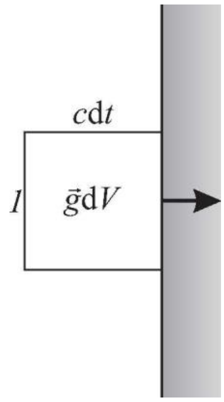


Figure 2.3: Radiation pressure at perpendicular incidence

Let the wave incidence be perpendicular to the surface of the body. The field in a cylinder with a base $1\mathrm{m}^2$ and a height $c\mathrm{d}t$ will be absorbed in the body surface unit in time $\mathrm{d}t$. This corresponds to the momentum transferred from the field to the body over time $\mathrm{d}t$, which equals

$$\mathrm{d}\boldsymbol{G} = \boldsymbol{g}\ \mathrm{d}V = \frac{w}{c}\ 1 \cdot c\ \mathrm{d}t\ \boldsymbol{s} = w\mathrm{d}t\ \boldsymbol{s}.$$

The corresponding force acting on the absorbing surface unit of the body is equal to the momentum transmitted per unit time, and the radiation pressure p is the magnitude of this force,

$$p = \left|\frac{\mathrm{d}\boldsymbol{G}}{\mathrm{d}t}\right| = w, \tag{2.18}$$

thus, *the radiation pressure is equal to the energy density in a plane wave.* If the surface of the body perfectly reflects the electromagnetic wave, the reflected wave has the same amplitude as the incident wave and thus the same momentum, but its direction is opposite. Therefore, momentum change and pressure have double value. If the *reflectivity* is *R*, then

$$p = (1 + \mathcal{R})w. \tag{2.19}$$

However, if the incidence of an electromagnetic plane wave on the body is not perpendicular, but has the general incidence angle ϑ measured from the normal to the surface, in order to determine the radiation pressure we need to use Maxwell stress tensor for a plane wave. Its components $(\sigma_{i1}, \sigma_{i2}, \sigma_{i3})$ constitute the flux density vector of the *i*-th component of the momentum and the force acting on the surface of the body is determined by the resultant flux of the momentum through the surface of the body.

Let the electromagnetic planar wave propagate in vacuum in the direction $\boldsymbol{s}$. Let us have the axes x_1, x_2, x_3 of the Cartesian coordinate system *S* directed to vectors $\boldsymbol{s}$, $\boldsymbol{E}$, $\boldsymbol{H}$ (in this order), i.e.

$$\boldsymbol{s} = (1, 0, 0), \quad \boldsymbol{E} = (0, E, 0), \quad \boldsymbol{H} = (0, 0, H). \tag{2.20}$$

Maxwell stress tensor of this wave

$$\sigma_{ij} = -[\varepsilon_0 E_i E_j + \mu_0 H_i H_j - \frac{1}{2}\delta_{ij}(\varepsilon_0 \boldsymbol{E}^2 + \mu_0 \boldsymbol{H}^2)] \tag{2.21}$$

then has all its non-diagonal elements equal to zero. If we use the relation $\sqrt{\varepsilon_0}E = \sqrt{\mu_0}H$, we obtain a diagonal matrix

$$(\sigma_{ij}) = \begin{pmatrix} w & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.22}$$

Since, according to (2.17), the momentum density of the wave for $\boldsymbol{s} = (1,0,0)$ is equal

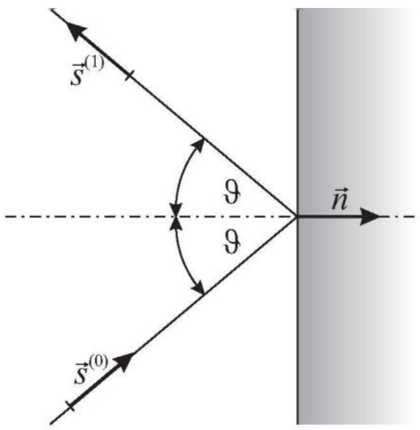
$$\boldsymbol{g} = \left(\frac{w}{c}, 0, 0\right), \tag{2.23}$$

we see that only the g_1 component that is transmitted with the density of the flux is nonzero

$$(\sigma_{11}, \sigma_{12}, \sigma_{13}) = (w, 0, 0) = g_1 c\boldsymbol{s}. \tag{2.24}$$

Since the other components σ_{ij} and g_i are zero, for this plane wave relation we can write the relation

$$(\sigma_{i1}, \sigma_{i2}, \sigma_{i3}) = g_i c\boldsymbol{s}. \tag{2.25}$$



Now, in order to calculate the force induced by a plane wave incident on a plane wall at any angle θ (Fig. 8.4), we need the components of Maxwell stress tensor in the generally oriented Cartesian system *S′*,

$$x_i' = c_{ij} x_j. \tag{2.26}$$

In the system *S′* then it is valid that

$$s_i' = c_{ij} s_j, \quad \sigma_{ij}' = c_{ik} c_{jl} \sigma_{kl} = c_{i1} c_{j1} w, \tag{2.27}$$

that is, for a wave propagating in a general direction $\boldsymbol{s}'$

$$\sigma_{ij}' = w s_i' s_j'. \tag{2.28}$$

The desired area force $\boldsymbol{T}$ acting on the wall surface unit is equal to the momentum flux through the wall surface unit,

$$\mathcal{T}_i = \sigma_{ij}^{(0)} n_j \cdot 1 + \sigma_{ij}^{(1)} n_j \cdot 1, \tag{2.29}$$

where $\sigma_{ij}^{(0)}$, $\sigma_{ij}^{(1)}$ are the components of Maxwell stress tensor of the incident and reflected wave, n is the unit vector of the normal to the wall (in the direction of the body). In the system fixedly connected to the wall according to equation (2.28)

$$\sigma_{ij}^{(0)} = w^{(0)} s'^{(0)}_i s'^{(0)}_j, \qquad \sigma_{ij}^{(1)} = w^{(1)} s'^{(1)}_i s'^{(1)}_j, \tag{2.30}$$

where unit vectors $\boldsymbol{s}'^{(0)}$, $\boldsymbol{s}'^{(i)}$ determine the propagation directions of the incident and reflected waves, therefore

$$\mathcal{T}_i = w^{(0)} s'^{(0)}_i s'^{(0)}_j n_j + w^{(1)} s'^{(1)}_i s'^{(1)}_j n_j. \tag{2.31}$$

Using the reflectivity $R = w^{(1)}/w^{(0)}$

$$\boldsymbol{\mathcal{T}} = w^{(0)} [\boldsymbol{s}'^{(0)} (\boldsymbol{s}'^{(0)} \cdot \boldsymbol{n}) + \mathcal{R} \boldsymbol{s}'^{(1)} (\boldsymbol{s}'^{(1)} \cdot \boldsymbol{n})], \tag{2.32}$$

where, according to the law of reflection $\boldsymbol{s}'^{(0)} \cdot \boldsymbol{n} = -\boldsymbol{s}'^{(1)} \cdot \boldsymbol{n} = \cos \vartheta$. *Radiation pressure* is the normal component of this force

$$p = \boldsymbol{\mathcal{T}} \cdot \boldsymbol{n} = w^{(0)} [(\boldsymbol{s}'^{(0)} \cdot \boldsymbol{n})^2 + \mathcal{R} (\boldsymbol{s}'^{(1)} \cdot \boldsymbol{n})^2] = (1 + \mathcal{R}) w^{(0)} \cos^2 \vartheta. \tag{2.33}$$

The tangent component of area force is

$$\mathcal{T}_t = (1 - \mathcal{R}) w^{(0)} \sin\vartheta \cos\vartheta. \qquad (2.34)$$

In 1899, a leading Russian physicist Pyotr Nikolaevich Lebedev (1866-1913) experimentally investigated the pressure effect of the beam of light that caused the deflection of lycopodium particles falling in an evacuated container (the drug lycopodium is spruce spores). Radiation pressure is also an important factor in astrophysics, since it affects the shape of the comet tails, and, being an important thermodynamic quantity, plays a role in the internal dynamics of stars.

The formula for the pressure P of the equilibrium radiation enclosed in a cavity is derived based on the assumption that radiation from all directions is incident on a given surface with equal probability. Radiation pressure P can be equivalently calculated by averaging the normal force components T for a fixed plane wave through all possible spatial orientations of the surface, i.e. through all directions of its normal n,

$$P = \langle \boldsymbol{T} \cdot \boldsymbol{n} \rangle_{\vartheta,\varphi} = \sigma_{ij} \langle n_i n_j \rangle_{\vartheta,\varphi}. \qquad (2.35)$$

For $i \neq j$ je $\langle n_i n_j \rangle_{\vartheta,\phi} = 0$, since n_j attains both positive and negative values with equal probability at a given i. Mean values $\langle n^2_i \rangle_{\vartheta,\phi}$ are all the same and their sum is equal to one. Therefore

$$\langle n_i n_j \rangle_{\vartheta,\varphi} = \frac{1}{3} \delta_{ij} \qquad (2.36)$$

and the pressure of P equilibrium radiation is equal to a third of the tensor trace σ_{ij}

$$P = \frac{1}{3} \sigma_{ij} \delta_{ij} = \frac{u}{3}, \qquad (2.37)$$

where u indicates the energy density of equilibrium radiation. However, the relation (2.36) can be equally obtained by laborious calculation of mean values over angles ϑ, φ in spherical coordinates,

$$\langle n_i n_j \rangle_{\vartheta,\varphi} = \frac{\int\limits_0^{2\pi} \int\limits_0^{\pi} n_i(\vartheta,\varphi) n_j(\vartheta,\varphi) \sin\vartheta \mathrm{d}\vartheta \mathrm{d}\varphi}{\int\limits_0^{2\pi} \int\limits_0^{\pi} \sin\vartheta \mathrm{d}\vartheta \mathrm{d}\varphi}.$$

Using the result (2.37), L. Boltzmann theoretically derived the law $u = \sigma T^4$ (T is the thermodynamic temperature) experimentally found by J. Stefan.

2.2 Monochromatic plane waves

The described d'Alembert solution of the wave equation is sufficiently general. Due to the linearity of the wave equation, the principle of superposition applies, i.e. the solution is again the the linear combination of two solutions of the wave equation. Therefore, any solution can be decomposed into a superposition of suitable simple wave types. The sophisticated mathematical theory of Fourier analysis offers *monochromatic (harmonic) plane waves* as the simplest components of linear decompositions.

The vector potential $\boldsymbol{A}(\boldsymbol{r}, t)$ of a *monochromatic plane wave* has a harmonic dependence on the phase $\boldsymbol{\xi} = \boldsymbol{s}.\boldsymbol{r} - vt$,

$$\boldsymbol{A}(\xi) = \boldsymbol{A}_0 \cos(k\xi + \varphi),$$

where φ is the *phase constant.* If $\boldsymbol{k} = k\boldsymbol{s}$ and $\omega = vk$, the vector potential of a monochromatic plane wave can be written in the form

$$\boldsymbol{A}(\boldsymbol{r}, t) = \boldsymbol{A}_0 \cos(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t + \varphi). \qquad (2.38)$$

$\mathbf{A}_0$ is called *wave amplitude,* ω *is the* … *is the wave number. Dispersion relation*

$$\omega = vk \quad \Leftrightarrow \quad \lambda = vT \quad \Leftrightarrow \quad \lambda\nu = v \qquad (2.39)$$

expresses the connection of time and s … *avelength* $\lambda = 2\pi/k$, *frequency* $\nu = \omega/2\pi = 1/T$ and *wavenumber* $k/2\pi = 1/\lambda$ indi…

It is often useful to consider a vector function (2.38) as a real part of a complex vector function

A(r, t) = Re A(r,t), where A(r,t)= $A_0 e^{i(k\,r - wi)}$ and $A_0 = A_0 e^{iV}$ (2.40)

$$\boldsymbol{A}(\boldsymbol{r}, t) = \mathrm{Re}\, \widehat{\boldsymbol{A}}(\boldsymbol{r}, t), \quad \text{kde} \quad \widehat{\boldsymbol{A}}(\boldsymbol{r}, t) = \widehat{\boldsymbol{A}}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} \quad \text{a} \quad \widehat{\boldsymbol{A}}_0 = \boldsymbol{A}_0 e^{i\varphi}$$

it is a complex vector amplitude. It is easy to see that the expression (2.40) is the solution to the wave equation (2.2). Since the wave equation (2.2) is linear and has real coefficients, the complex wave (2.40) is its solution just when the solutions are Re $\hat{\boldsymbol{A}}$ and Im $\hat{\boldsymbol{A}}$. If we only perform linear operations with real coefficients, we can safely work with a complex notation and finally proceed to the real part. For energy quantities that are quadratic, it is necessary to derive special formulas.

According to the expression (2.10), the complex vector potential (2.40) determines a complex field

$$\widehat{\boldsymbol{E}} = -\frac{\partial \widehat{\boldsymbol{A}}}{\partial t} = i\omega\widehat{\boldsymbol{A}}, \qquad \widehat{\boldsymbol{B}} = \mathrm{rot}\widehat{\boldsymbol{A}} = i\boldsymbol{k} \times \widehat{\boldsymbol{A}}. \tag{2.41}$$

At the same time, $\hat{\boldsymbol{A}}$ is limited by the condition (2.3),

$$\mathrm{div}\widehat{\boldsymbol{A}} = i\boldsymbol{k} \cdot \widehat{\boldsymbol{A}} = 0. \tag{2.42}$$

Relations (2.41), (2.42) express mutual orthogonality and proportions of vector magnitudes $\boldsymbol{k}$, $\hat{\boldsymbol{E}}$, $\widehat{\boldsymbol{B}}$. First of all, it is apparent that the magnetic field is fully determined by the electric field

$$\widehat{\boldsymbol{E}}(\boldsymbol{r}, t) = \widehat{\boldsymbol{E}}_0 \; e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}, \tag{2.43}$$

whose complex amplitude $\hat{\boldsymbol{E}}_0$ is orthogonal to the direction of propagation $\boldsymbol{k}$.

Therefore, let us consider the electric field of the monochromatic plane wave in more detail. For certainty, let us search for the geometric location of the endpoints of the vector $\boldsymbol{E}(\boldsymbol{r}, t)$ = Re $\hat{\boldsymbol{E}}(\boldsymbol{r}, t)$ at a given point in space $\boldsymbol{r}$. We will select the coordinate system with the axis z in the direction $\boldsymbol{k}$ and express the complex components of the vector $\hat{\boldsymbol{E}}_0$ in the form

$$\widehat{E}_{0x} = E_1 e^{i\varphi_1}, \quad \widehat{E}_{0y} = E_2 e^{i\varphi_2}, \quad \widehat{E}_{0z} = 0. \tag{2.44}$$

Then

$$\boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{x}_0 E_1 \cos(kz - \omega t + \varphi_1) + \boldsymbol{y}_0 E_2 \cos(kz - \omega t + \varphi_2), \tag{2.45}$$

where x_0, y_0 are unit vectors in the directions of axes x, y. The expression (2.45) represents the general form of a monochromatic plane wave. It depends on four arbitrary constants E_1, E_2, φ_1, φ_2, or two complex numbers (2.44). In order to express $\boldsymbol{E}(\boldsymbol{r}, t)$ more clearly, let us demonstrate that the complex vector $\hat{\boldsymbol{E}}_0$ can always be written in the form

$$\widehat{\boldsymbol{E}}_0 = (\boldsymbol{b}_1 + i\boldsymbol{b}_2)e^{-i\beta}, \tag{2.46}$$

where $\boldsymbol{b}_1$, $\boldsymbol{b}_2$ are real vectors, $\boldsymbol{b}_1 \cdot \boldsymbol{b}_2 = 0$ and phase $\beta \in \mathbb{R}$ is determined from its square. The $\hat{\boldsymbol{E}}_0$ vector square is generally a complex number that we write with the phase -2β,

$$(\widehat{\boldsymbol{E}}_0)^2 = \widehat{\boldsymbol{E}}_0 \cdot \widehat{\boldsymbol{E}}_0 = |(\widehat{\boldsymbol{E}}_0)^2| e^{-2i\beta};$$

then the complex vector $\boldsymbol{b}_1 + i\boldsymbol{b}_2$ defined by the expression (2.46) will have a real square

$$(\boldsymbol{b}_1 + i\boldsymbol{b}_2)^2 = \boldsymbol{b}_1^2 - \boldsymbol{b}_2^2 + 2i\boldsymbol{b}_1 \cdot \boldsymbol{b}_2 = (\widehat{\boldsymbol{E}}_0)^2 e^{2i\beta} = |(\widehat{\boldsymbol{E}}_0)^2|,$$

from where $\boldsymbol{b}_1 \cdot \boldsymbol{b}_2 = 0$. This proved the relation (2.46).

Now let us select the axis z in the direction $\boldsymbol{k}$ and the axis x in the direction $\boldsymbol{b}_1$. Then the relations (2.43) and (2.46) yield

$$\boldsymbol{E}(\boldsymbol{r}, t) = \mathrm{Re}\{(\boldsymbol{b}_1 + i\boldsymbol{b}_2)e^{i(kz - \omega t - \beta)}\} = \boldsymbol{b}_1 \cos(kz - \omega t - \beta) - \boldsymbol{b}_2 \sin(kz - \omega t - \beta). \tag{2.47}$$

In the components

$$\begin{aligned} E_x &= b_1 \cos(\omega t - kz + \beta), \\ E_y &= \pm b_2 \sin(\omega t - kz + \beta), \\ E_z &= 0, \end{aligned} \tag{2.48}$$

where $\boldsymbol{b}_1 = b_1\boldsymbol{x}_0$, $\boldsymbol{b}_2 = b_2\boldsymbol{y}_0$, b_1, $b_2 \geq 0$. In the xy plane, the following equations apply

$$\frac{E_x^2}{b_1^2} + \frac{E_y^2}{b_2^2} = 1. \tag{2.49}$$

This result geometrically means that at each point of space, the vector $\boldsymbol{E}$ of a monochromatic plane wave rotates in the plane perpendicular to $\boldsymbol{k}$ with its end following an elliptic trajectory: *a monochromatic plane wave is in general elliptically polarized.* The special case $b_1 = b_2$ is referred to as *circular polarization*, the case $b_1 = 0$ or $b_2 = 0$ is *linear polarization.* According to the vector $\boldsymbol{E}$ rotation direction (upper or lower sign of E_y in (2.48)), we distinguish between *the right-handed or left-handed circular polarization.*

If the energy quantity in a monochromatic wave quickly oscillate with a period T, we are concerned with their *time mean values*, which for periodic functions $f(t + T) = f(t)$ are defined by the integration over a time interval T,

$$\langle f(t)\rangle_T = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\mathrm{d}t. \tag{2.50}$$

Using a standard formula

$$\langle\cos^2(\omega t + \delta)\rangle_T = \frac{1}{2}$$

then we derive for a linearly polarized wave

$$\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}_0\cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t + \alpha) \tag{2.51}$$

time mean values

$$\begin{aligned}\langle w\rangle_T &= \varepsilon\langle E^2\rangle_T = \frac{1}{2}\varepsilon E_0^2,\\ \langle \boldsymbol{S}\rangle_T &= \langle w\rangle_T v\boldsymbol{s} = \frac{1}{2Z}E_0^2\boldsymbol{s},\\ \langle \boldsymbol{g}\rangle_T &= \frac{\langle \boldsymbol{S}\rangle_T}{c^2} = \frac{\langle w\rangle_T}{c}\boldsymbol{s}.\end{aligned}$$

For the general case of elliptically polarized wave (2.43) (in the complex notation), the following formulas apply

$$\langle w\rangle_T = \varepsilon\langle(\mathrm{Re}\widehat{\boldsymbol{E}})^2\rangle_T = \frac{1}{2}\varepsilon(\widehat{\boldsymbol{E}}\cdot\widehat{\boldsymbol{E}}^*) = \frac{1}{2}\varepsilon|\widehat{\boldsymbol{E}}_0|^2, \tag{2.55}$$

$$\langle \boldsymbol{S}\rangle_T = \frac{1}{2Z}|\widehat{\boldsymbol{E}}_0|^2\boldsymbol{s}, \tag{2.56}$$

and (2.54). The asterisk here indicates a complex conjugate vector and a square of the module defined in the usual way

$$|\widehat{\boldsymbol{E}}_0|^2 = |\widehat{E}_{0x}|^2 + |\widehat{E}_{0y}|^2 + |\widehat{E}_{0z}|^2 = \widehat{\boldsymbol{E}}_0\cdot\widehat{\boldsymbol{E}}_0^*$$

is equal for (2.45) or (2.48)

$$|\widehat{\boldsymbol{E}}_0|^2 = E_1^2 + E_2^2 = b_1^2 + b_2^2.$$

2.3 Monochromatic plane wave on a boundary

When the electromagnetic wave is incident on the boundary of two transparent media, both their reflection and refraction occur. The relations between the amplitudes of individual waves are determined by *Fresnel formulae.* To derive them, we need to apply the conditions on the boundary of non-conductive media, which we will now derive.

Conditions on the boundary of non-conductive media

Maxwell's equations can be solved in a homogeneous isotropic medium with material constants ε, μ. However, there are often situations, especially in optics, when the values of constants ε, μ suddenly change on the boundaries of two homogeneous media. Assuming the validity of the relation $\boldsymbol{D} = \varepsilon\boldsymbol{E}$, at least one of the vectors $\boldsymbol{E}$, $\boldsymbol{D}$ must be discontinuous and therefore we cannot use Maxwell's equations on the boundary. We will derive the *conditions on the boundary* from the integral form of Maxwell's equations based on the notion of a discontinuous change of constants on the boundary as the limit of continuous transition.

The behavior of *electric induction* $\boldsymbol{D}$ on the boundary is derived from the Maxwell's equation of the div$\boldsymbol{D} = \rho$ written in the integral form

$$\oint_{\partial V}\boldsymbol{D}\cdot\mathrm{d}\boldsymbol{f} = \int_V \rho\mathrm{d}V.$$

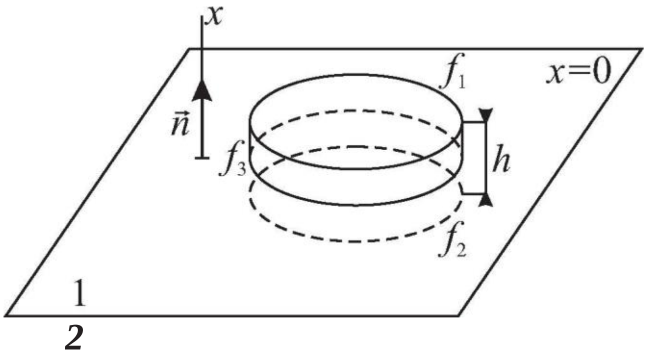


Figure 2.5: Electric induction on the boundary

Let us assume that the volume V has, according to Figure 8.5, the shape of a cylinder of height h, whose surface ∂V consists of a base f_1, f_2 in media 1, 2 and side f_3. Then we have

$$\int_{f_1} \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{f} + \int_{f_2} \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{f} + \int_{f_3} \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{f} = \int_V \rho \mathrm{d}V. \quad (2.57)$$

In the limit $h \to 0$, the integral through f_3 disappears, $\lim_{h\to 0} \rho h = \sigma$, and there remains

$$\int_f (\boldsymbol{D}_1 - \boldsymbol{D}_2) \cdot \boldsymbol{n} \mathrm{d}f = \int_f \sigma \mathrm{d}f, \quad (2.58)$$

where n is the normal to the boundary pointing into the medium 1, f is the cross-section of the cylinder with the boundary, $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ are the limits of the vector $\boldsymbol{D}$ from the medium 1 and 2, and σ is the surface density of free charge on the boundary. Since the surface f was arbitrary,

$$\mathrm{Div}\boldsymbol{D} = \boldsymbol{n} \cdot (\boldsymbol{D}_1 - \boldsymbol{D}_2) = \sigma. \quad (2.59)$$

The formal operation Div is referred to as *surface divergence*. Note that if the boundary is a plane $x = 0$, the surface charge on this plane is described by the singular volume density

$$\rho(x, y, z) = \delta(x)\sigma(y, z), \quad (2.60)$$

therefore, the total charge

$$\int_V \rho \mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_f \sigma \mathrm{d}y\mathrm{d}z. \quad (2.61)$$

The behavior of *magnetic induction* $\boldsymbol{B}$ is obtained by the same procedure from the Maxwell's equation div$\boldsymbol{B}$ = 0:

$$\lim_{h\to 0} \oint_{\partial V} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{f} = \int_f (\boldsymbol{B}_1 - \boldsymbol{B}_2) \cdot \boldsymbol{n} \mathrm{d}f = 0 \quad (2.62)$$

and therefore

$$\mathrm{Div}\boldsymbol{B} = \boldsymbol{n} \cdot (\boldsymbol{B}_1 - \boldsymbol{B}_2) = 0. \quad (2.63)$$

The behavior of the *magnetic intensity* $\boldsymbol{H}$ on the boundary is derived from the Maxwell's equation rot$\boldsymbol{H} - (\partial \boldsymbol{D}/\partial t) = j$ written in the integral form,

$$\oint_{\partial f} \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} - \int_f \frac{\partial \boldsymbol{D}}{\partial t} \cdot \mathrm{d}\boldsymbol{f} = \int_f \boldsymbol{j} \cdot \mathrm{d}\boldsymbol{f}. \quad (2.64)$$

According to Fig. 8.6, the surface f is a rectangle of length l along the boundary and height h perpendicular to the boundary, with a normal N lying in the plane of the boundary. Its circumference $\partial f = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$ is oriented in accordance with normal and consists of four sides, with Γ_1 and Γ_2 lying in media 1 and 2, Γ_3, Γ_4 connecting the two media. If we perform the limit

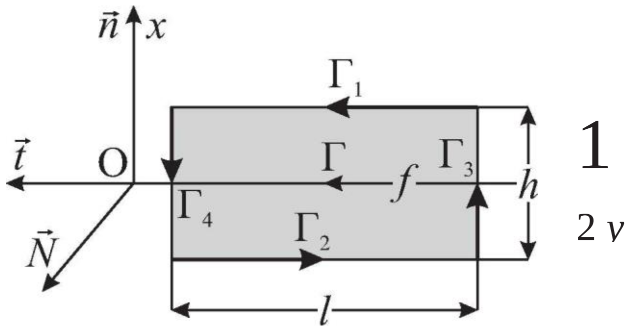


Figure 2.6: the intensity of the magnetic field on the boundary

limit h → 0 in the equation (2.64), the term with the time derivation disappears (provided that the integrand is limited everywhere). The integrals through Γ_1 and Γ_2 will be on the left side, and the possible surface current through the surface *f* with a surface density *i* will be on the right side:

$$\int_\Gamma (\boldsymbol{H}_1 - \boldsymbol{H}_2)\cdot \boldsymbol{t}\mathrm{d}l = \int_\Gamma \boldsymbol{i}\cdot \boldsymbol{N}\mathrm{d}l. \tag{2.65}$$

Here we used $\lim_{h\to 0} \boldsymbol{j}.\boldsymbol{N}h = i \cdot \boldsymbol{N}$ and introduced a unit vector $\boldsymbol{t} = \boldsymbol{N} \times \boldsymbol{n}$, therefore

$$(\boldsymbol{H}_1 - \boldsymbol{H}_2)\cdot \boldsymbol{t} = \boldsymbol{n} \times (\boldsymbol{H}_1 - \boldsymbol{H}_2)\cdot \boldsymbol{N}. \tag{2.66}$$

Since the curve Γ (the intersection of the surface *f* with the boundary) was arbitrary, the relations (2.65) and (2.66) yield

$$\boldsymbol{n} \times (\boldsymbol{H}_1 - \boldsymbol{H}_2)\cdot \boldsymbol{N} = \boldsymbol{i}\cdot \boldsymbol{N}. \tag{2.67}$$

Now the vectors $\boldsymbol{n} \times (\boldsymbol{H}_1 - \boldsymbol{H}_2)$, $\boldsymbol{i}$ and $\boldsymbol{N}$ lie in the plane of the boundary and because $\boldsymbol{N}$ has an arbitrary direction in this plane, it is valid that

$$\mathrm{Rot}\boldsymbol{H} = \boldsymbol{n} \times (\boldsymbol{H}_1 - \boldsymbol{H}_2) = \boldsymbol{i}. \tag{2.68}$$

A formal operation Rot is referred to as a *surface rotation.* Note that when choosing the axes *x, y, z* in directions $\boldsymbol{n}$, $-\boldsymbol{t}$, $-\boldsymbol{N}$, the surface current on the boundary is described by the singular volume density

$$\boldsymbol{j}(x, y, 0) = \delta(x)\boldsymbol{i}(y, 0). \tag{2.69}$$

Then the current through the surface *f* is

$$\int_f \boldsymbol{j}\cdot \boldsymbol{N}\mathrm{d}x\mathrm{d}y = \int_\Gamma \boldsymbol{i}\cdot \boldsymbol{N}\mathrm{d}y. \tag{2.70}$$

The *electric intensity* $\boldsymbol{E}$ behavior on the boundary follows the same limit procedure from the Maxwell's equation rot$\boldsymbol{E}$ $(\partial B/\partial t) = 0$ written in the integral form,

$$\lim_{h\to 0}\left[\oint_{\partial f} \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{l} + \int_f \frac{\partial B}{\partial t}\cdot \mathrm{d}\boldsymbol{f}\right] = \int_\Gamma (\boldsymbol{E}_1 - \boldsymbol{E}_2)\cdot \boldsymbol{t}\mathrm{d}l = 0, \tag{2.71}$$

and therefore

$$(\boldsymbol{E}_1 - \boldsymbol{E}_2)\cdot \boldsymbol{t} = E_{1t} - E_{2t} = 0, \tag{2.72}$$

or in the vector form

$$\mathrm{Rot}\boldsymbol{E} = \boldsymbol{n} \times (\boldsymbol{E}_1 - \boldsymbol{E}_2) = 0. \tag{2.73}$$

Fresnel formulae

Let us consider the plane boundary of two media 1, 2, with material constants ε_1, μ_1, ε_2, μ_2, on which a monochromatic plane wave $\boldsymbol{E}_0(\boldsymbol{r}, t)$ is incident with frequency ω_0 and wave vector $\boldsymbol{k}_0$. According to figure 8.7, we denote the quantities related to waves as incident, reflected, and transmitted with the indices $i = 0, 1, 2$, respectively. The respective angles ϑ_i are measured from the normal to the boundary characterized by unit vector $\boldsymbol{n}$ pointing to the medium 1. We will further consider only linearly polarized waves, for which the vectors $\boldsymbol{E}_i(\boldsymbol{r}, t)$ are real, so they can be written in the form

$$\boldsymbol{E}_i(\boldsymbol{r}, t) = \boldsymbol{E}_i \cos(\boldsymbol{k}_i \cdot \boldsymbol{r} - \omega_i t).$$

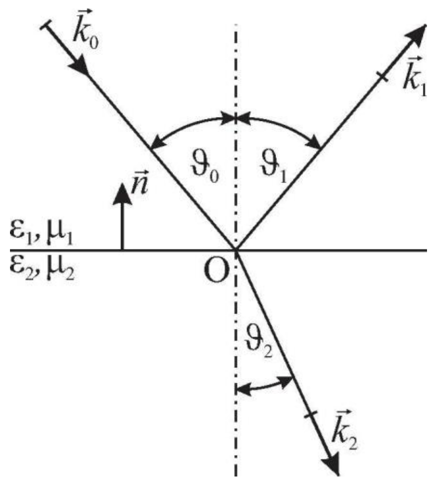


Figure 2.7: Plane electromagnetic wave on the boundary

The plane of incidence ρ is determined by the vector ko and the vector of the normal n. The problem clearly has *translation symmetry in the plane of the boundary* and it is sufficient to solve it in one plane ρ. This translational symmetry also implies that the reflected and transmitted waves will be planar. The further *symmetry of the problem with respect to reflection by the plane* ρ implies that vectors $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ will lie in the plane ρ together with $\boldsymbol{k}_0$.

If we use a complex notation

$$\widehat{\boldsymbol{E}}_i(\boldsymbol{r}, t) = \boldsymbol{E}_i e^{i(\boldsymbol{k}_i \cdot \boldsymbol{r} - \omega_i t)}, \quad (i = 0, 1, 2), \tag{2.74}$$

the condition (2.72) of continuity of tangent components $\boldsymbol{E}$ reads

$$E_{0t} e^{i(\boldsymbol{k}_0 \cdot \boldsymbol{r} - \omega_0 t)} + E_{1t} e^{i(\boldsymbol{k}_1 \cdot \boldsymbol{r} - \omega_1 t)} = E_{2t} e^{i(\boldsymbol{k}_2 \cdot \boldsymbol{r} - \omega_2 t)} \tag{2.75}$$

and must apply for any time t and at all points $\boldsymbol{r}$ of the boundary. The linear independence of exponential functions yields

$$\omega_0 = \omega_1 = \omega_2 \tag{2.76}$$

and

$$k_{0t} = k_{1t} = k_{2t} \quad \Leftrightarrow \quad k_0 \sin\vartheta_0 = k_1 \sin\vartheta_1 = k_2 \sin\vartheta_2. \tag{2.77}$$

Hence, using dispersion relations

$$\omega_0 = v_1 k_0, \quad \omega_1 = v_1 k_1, \quad \omega_2 = v_2 k_2, \tag{2.78}$$

where

$$v_1 = \frac{1}{\sqrt{\varepsilon_1 \mu_1}} = \frac{c}{n_1}, \qquad v_2 = \frac{1}{\sqrt{\varepsilon_2 \mu_2}} = \frac{c}{n_2}, \tag{2.79}$$

we obtain important 'kinematic' relations for reflection and refraction: *the law of reflection*

$$\vartheta_0 = \vartheta_1, \tag{2.80}$$

• *Snell's law of refraction*

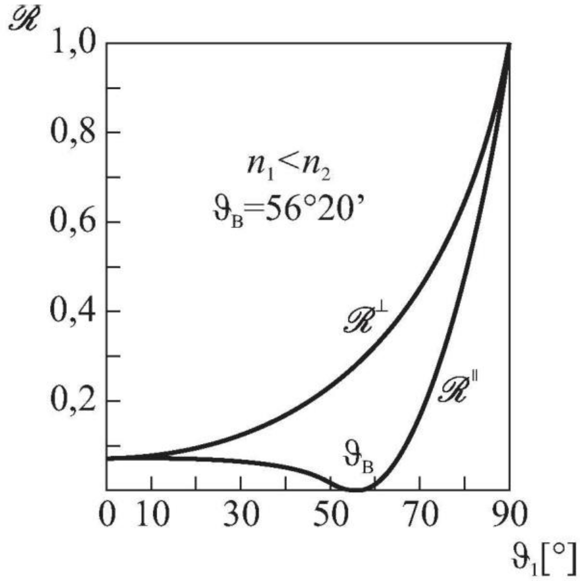
$$\frac{\sin\vartheta_1}{\sin\vartheta_2} = \frac{k_2}{k_1} = \frac{v_1}{v_2} = \frac{n_2}{n_1} = \sqrt{\frac{\varepsilon_2\mu_2}{\varepsilon_1\mu_1}}. \tag{2.81}$$

We will further use only angles ϑ_1 and ϑ_2.

Maxwell's theory allows further determination of 'dynamic' relations─the intensity ratios of individual waves depending on the angle of incidence and polarization. Since all the exponentials in the relation (2.75) are equal on the boundary, we can express the conditions on the boundary from relations (2.59), (2.73), (2.63), (2.68) using the amplitudes E_i:

$$\boldsymbol{n}\cdot[\varepsilon_1(\boldsymbol{E}_0+\boldsymbol{E}_1)-\varepsilon_2\boldsymbol{E}_2]=0,$$
$$\boldsymbol{n}\times(\boldsymbol{E}_0+\boldsymbol{E}_1-\boldsymbol{E}_2)=0,$$
$$\boldsymbol{n}\cdot[\sqrt{\varepsilon_1\mu_1}(\boldsymbol{s}_0\times\boldsymbol{E}_0+\boldsymbol{s}_1\times\boldsymbol{E}_1)-\sqrt{\varepsilon_2\mu_2}\boldsymbol{s}_2\times\boldsymbol{E}_2]=0,$$
$$\boldsymbol{n}\times\left[\sqrt{\frac{\varepsilon_1}{\mu_1}}(\boldsymbol{s}_0\times\boldsymbol{E}_0+\boldsymbol{s}_1\times\boldsymbol{E}_1)-\sqrt{\frac{\varepsilon_2}{\mu_2}}\boldsymbol{s}_2\times\boldsymbol{E}_2\right]=0.$$

Since each monochromatic wave can be decomposed into the superposition of two linearly polarized waves with orthogonal directions of polarization, it is sufficient to derive the Fresnel formulae for two cases, where the polarization direction is perpendicular to the plane of incidence ρ or is parallel to that plane.



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Case E0 ⊥ *ρ*. The symmetry with respect to the reflection by the plane ρ implies that $E_i \perp \rho$, i = 0,1, 2. The condition (2.82) is fulfilled identically, since all electric vectors are parallel to the boundary, $\boldsymbol{n}\cdot\boldsymbol{E}_i = 0$. After dividing E_0, the condition (2.83) yields

$$\frac{E_2}{E_0} = 1 + \frac{E_1}{E_0}. \tag{2.86}$$

The relation (2.84) yields the same equation flows with respect to (2.81). The condition (2.85) remains, which, due to relations

$$\boldsymbol{n}\times(\boldsymbol{s}_i\times\boldsymbol{E}_i) = -E_i(\boldsymbol{n}\cdot\boldsymbol{s}_i), \quad \boldsymbol{n}\cdot\boldsymbol{s}_0 = -\boldsymbol{n}\cdot\boldsymbol{s}_1 = -\cos\vartheta_1, \quad \boldsymbol{n}\cdot\boldsymbol{s}_2 = -\cos\vartheta_2$$

after dividing E_0 yields

$$\sqrt{\frac{\varepsilon_1}{\mu_1}}\left(-1+\frac{E_1}{E_0}\right)\cos\vartheta_1 + \sqrt{\frac{\varepsilon_2}{\mu_2}}\frac{E_2}{E_0}\cos\vartheta_2 = 0. \tag{2.87}$$

The solution of the system of linear equations (2.86), (2.87) is the *reflection coefficient for the amplitude*

$$R_E^{\perp} = \frac{E_1}{E_0} = \frac{\sqrt{\frac{\varepsilon_1}{\mu_1}}\cos\vartheta_1 - \sqrt{\frac{\varepsilon_2}{\mu_2}}\cos\vartheta_2}{\sqrt{\frac{\varepsilon_1}{\mu_1}}\cos\vartheta_1 + \sqrt{\frac{\varepsilon_2}{\mu_2}}\cos\vartheta_2} = \frac{1-\frac{\mu_1 \mathrm{tg}\vartheta_1}{\mu_2 \mathrm{tg}\vartheta_2}}{1+\frac{\mu_1 \mathrm{tg}\vartheta_1}{\mu_2 \mathrm{tg}\vartheta_2}} \tag{2.88}$$

and *the permeability coefficient for the amplitude*

$$T_E^{\perp} = \frac{E_2}{E_0} = 1 + R_E^{\perp} = \frac{2}{1 + \frac{\mu_1 \mathrm{tg}\vartheta_1}{\mu_2 \mathrm{tg}\vartheta_2}}. \tag{2.89}$$

They also can be written using the characteristic impedance $Z_i = \sqrt{\mu_i/\varepsilon_i}$. In the case that $\mu_1 = \mu_2$, which is usually fulfilled at optical frequencies, the *Fresnel formulae* (Augustin Jean Fresnel, 1788-1827) are valid

$$R_E^{\perp} = \frac{\sin(\vartheta_2 - \vartheta_1)}{\sin(\vartheta_2 + \vartheta_1)}, \quad T_E^{\perp} = \frac{2\cos\vartheta_1 \sin\vartheta_2}{\sin(\vartheta_1 + \vartheta_2)}. \tag{2.90}$$

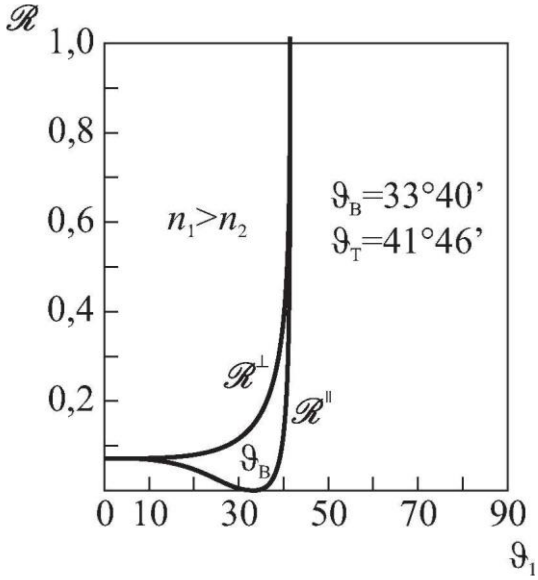

Figure 2.9: reflectivity for the transition from glass to air

Case $\boldsymbol{E}_0 \parallel \rho$. Here we will mention only the results:

$$R_E^{\parallel} = \frac{\mu_1 \sin 2\vartheta_1 - \mu_2 \sin 2\vartheta_2}{\mu_1 \sin 2\vartheta_1 + \mu_2 \sin 2\vartheta_2}, \quad T_E^{\parallel} = \sqrt{\frac{\varepsilon_1 \mu_1}{\varepsilon_2 \mu_2}} \frac{2\mu_2 \sin 2\vartheta_1}{\mu_1 \sin 2\vartheta_1 + \mu_2 \sin 2\vartheta_2} \tag{2.91}$$

and for $\mu_1 = \mu_2$

$$R_E^{\parallel} = \frac{\mathrm{tg}(\vartheta_1 - \vartheta_2)}{\mathrm{tg}(\vartheta_1 + \vartheta_2)}, \quad T_E^{\parallel} = \frac{2\cos\vartheta_1 \sin\vartheta_2}{\sin(\vartheta_1 + \vartheta_2)\cos(\vartheta_1 - \vartheta_2)}. \tag{2.92}$$

At *perpendicular incidence* ($\vartheta_1 = \vartheta_2 = 0$) both cases yield

$$R_E = \frac{Z_2 - Z_1}{Z_1 + Z_2}, \qquad T_E = 1 + R_E \tag{2.93}$$

and at $\mu_1 = \mu_2$ there are known formulae

$$R_E = \frac{n_1 - n_2}{n_1 + n_2}, \qquad T_E = \frac{2n_1}{n_1 + n_2}. \tag{2.94}$$

In practice, the amplitudes E_i are not measured, but the intensities

$$\mathcal{I}_i = |\langle \boldsymbol{S}_i \rangle_T| = \frac{1}{2Z_i} E_i^2.$$

Then the *reflectivity R* defined by the relation

$$\mathcal{R} = \frac{\mathcal{I}_1}{\mathcal{I}_0} = \frac{E_1^2}{E_0^2}, \tag{2.95}$$

is equal to the square of the reflection coefficient for the amplitude, and for $\mu_1 = \mu_2$

$$\mathcal{R}^{\perp} = \frac{\sin^2(\vartheta_1 - \vartheta_2)}{\sin^2(\vartheta_1 + \vartheta_2)}, \qquad \mathcal{R}^{\parallel} = \frac{\mathrm{tg}^2(\vartheta_1 - \vartheta_2)}{\mathrm{tg}^2(\vartheta_1 + \vartheta_2)}. \tag{2.96}$$

The graphs show the dependence of the reflectivity R on the angle of incidence $\vartheta_0 = \vartheta_1$ for the transition from air to glass (Fig.8.8) and from glass to air (Fig. 8.9); the values $n_1 = 1$ and $n_2 = 1.5$ are selected. The angle ϑ is *Brewster's angle* ($R^{|} = 0$ při $\vartheta_1 + \vartheta_2 = \pi/2$, therefore $\mathrm{tg}\vartheta_B = n_2/n_1$); ϑ_Γ is the limit angle, from which *total reflection* only occurs when $n_1 < n_2$ (when $\vartheta_1 = \vartheta_\Gamma$, $\vartheta_2 = \pi/2$, therefore $\sin \vartheta_\Gamma = n_2/n_1$).

2.4 Solution of non-homogeneous wave equations

We have so far only discussed the propagation of electromagnetic waves, not their formation. We shall see that the necessary condition for moving charged particles to induce electromagnetic radiation is nonzero acceleration. Now we will derive the potentials of the electromagnetic field induced by any specified *nonstationary* distribution of charges $\rho(\boldsymbol{r}, t)$ and currents j (($\boldsymbol{r}$, t). We will again assume a homogeneous isotropic medium at rest (with material constants ε, μ), for which Maxwell's equations apply in their usual form. These equations were equivalently replaced by a system of four inhomogeneous equations

$$\Delta\varphi - \frac{1}{v^2}\frac{\partial^2\varphi}{\partial t^2} = -\frac{\rho}{\varepsilon}, \tag{2.97}$$

$$\Delta \boldsymbol{A} - \frac{1}{v^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\mu \boldsymbol{j}, \tag{2.98}$$

for electromagnetic potentials φ, $\boldsymbol{A}$, which meet the Lorenz gauge condition

$$\mathrm{div}\ \boldsymbol{A} + \frac{1}{v^2}\frac{\partial\varphi}{\partial t} = 0. \tag{2.99}$$

We have already used Maxwell's formula $v^2 = 1/\varepsilon\mu$ to write these equations.

The general solution of each of the four partial differential equations (2.97), (2.98) consists of a general solution of the homogeneous equation and a particular solution of the non-homogeneous equation. The basic types of solutions of homogeneous equations were derived in the previous paragraphs. Since all four equations have the same mathematical form, we will confine ourselves to determining the particular solution of equation (2.97) for the scalar potential with the prescribed source $\rho(\boldsymbol{r}, t)$. First, we will formulate the problem of finding potential in electrostatics, in order to determine the potential in a non-stationary case in an analogous way.

The electrostatic potential

$$\varphi(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon}\int_V \frac{\rho(\boldsymbol{r}')\mathrm{d}V'}{|\boldsymbol{r} - \boldsymbol{r}'|}$$

is a particular solution of the Poisson's equation

$$\Delta\varphi = -\frac{\rho}{\varepsilon}, \tag{2.101}$$

which (at density ρ different from zero only in the finite region V) at infinity is going to zero at least as rapid as $1/r$:

$$|\lim_{r\to\infty} r\varphi(\boldsymbol{r})| < \infty. \tag{2.102}$$

The potential (2.100) can be viewed as the integral sum of the contributions of elementary Coulomb potentials at the observer location P, which are induced by charges $\rho(\boldsymbol{r}')\mathrm{d}V'$ in the volume elements $\mathrm{d}V'$ (Fig. 8.10).

It is sufficient to determine the elementary Coulomb potential for the point charge Q located in the origin. The solution to this problem is clearly determined by four requirements:

1. Laplace's equation $\Delta\varphi = 0$ for $r \neq 0$;

2. spherical symmetry $\varphi = \varphi(r)$;

3. boundary condition $\varphi \to 0$ for $r \to \infty$;

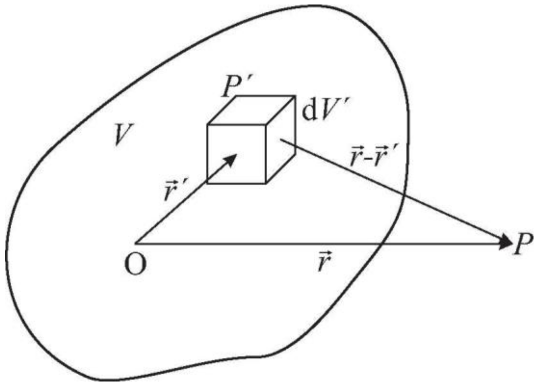

Figure 2.10: To the particular integral of the Poisson's equation

4. The Gauss theorem <*f* $\boldsymbol{E}.\mathrm{d}\boldsymbol{f} = Q/\varepsilon$ for a region containing the origin and bounded by a closed surface *f*.

The Laplace's equation for spherically symmetrical potential $\varphi(r)$

$$\Delta\varphi(r) = \frac{\mathrm{d}^2\varphi}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}\varphi}{\mathrm{d}r} = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\varphi}{\mathrm{d}r}\right) = 0 \quad (r \neq 0) \tag{2.103}$$

is easily integrated twice into

$$\varphi = \frac{B}{r} + A. \tag{2.104}$$

The integration constants are then unambiguously determined by conditions 3 and 4: $A = 0$, $B = Q/4\pi\varepsilon$. The resulting potential is then, as expected,

$$\varphi(r) = \frac{Q}{4\pi\varepsilon r}. \tag{2.105}$$

Let us now search for the elementary potential for a *non-stationary problem* (2.97) with a point charge $Q(t)$ located in the origin. We will require that the potential $\varphi(\boldsymbol{r}, t)$ satisfies the conditions

1. wave equation $\Delta\varphi - \frac{1}{v^2}\frac{\partial^2\varphi}{\partial t^2} = 0$ for $r \neq 0$;
2. spherical symmetry $\varphi = \varphi(r, t)$;
3. instead of a boundary condition, the requirement that within the limit $v \to \infty$, the potential transits into the instantaneous Coulomb potential

$$\varphi_\infty(\boldsymbol{r}, t) = \frac{Q(t)}{4\pi\varepsilon r}; \tag{2.106}$$

4. the condition of radiation, which will be further specified.

The wave equation for spherically symmetrical potential after substitution

$$\Delta\varphi(r, t) = \frac{\partial^2\varphi}{\partial r^2} + \frac{2}{r}\frac{\partial\varphi}{\partial r} = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\varphi) \tag{2.107}$$

and multiplying $r \neq 0$ we will transform into the form

$$\frac{\partial^2}{\partial r^2}(r\varphi) - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}(r\varphi) = 0. \tag{2.108}$$

This is however a wave equation for the function $r\varphi(r, t)$. Thus, its solution can be written as a superposition of two successive traveling waves

$$r\varphi(r, t) = f_R\left(t - \frac{r}{v}\right) + f_A\left(t + \frac{r}{v}\right), \tag{2.109}$$

$$\varphi(r, t) = \frac{f_R(t - \frac{r}{v})}{r} + \frac{f_A(t + \frac{r}{v})}{r}. \tag{2.110}$$

According to the third condition, arbitrary functions f_R, f_A are related to the time-varying charge in the origin by the relation

$$f_R(t) + f_A(t) = \frac{Q(t)}{4\pi\varepsilon}. \tag{2.111}$$

The first term of the potentials (2.110) can be interpreted as the field induced by the charge $4\pi\varepsilon f_R$ at a distance r and at time t according to the Coulomb formula, but for the charge we substitute the value that the charge had in the *retarded time* $t' = t -$ (r/v). Therefore, the charge $4\pi\varepsilon f_R(t)$ induces the Coulomb potential at a distance r, but delayed in time by r/v. Therefore, it is a *retarded potential.* The time r/v is the time interval that needs a signal propagating at speed v in order to cover the distance r. The changes in the charge in the origin induces a disturbance of the field that propagates as a *diverging spherical wave* at velocity v. On the contrary, the second term in the expression (2.110) represents *a converging spherical wave, an advanced potential.* Our problem is defined as the determination of the field induced by the charge $Q(t)$. According to the *principle of causality*, this corresponds to the retarded potential. Therefore, we impose the condition

4th radiation condition: $f_A(t) = 0$.

Thus, the potential of the variable point charge in the origin will be retarded,

$$\varphi(\boldsymbol{r}, t) = \frac{Q(t - \frac{r}{v})}{4\pi\varepsilon r}. \tag{2.112}$$

According to the principle of superposition by analogy with (2.100), for a charge distributed with density $\rho(\boldsymbol{r}, t)$ in the volume V we obtain *the retarded scalar potential*

$$\varphi(\boldsymbol{r}, t) = \frac{1}{4\pi\varepsilon} \int_V \frac{\rho(\boldsymbol{r}', t')\mathrm{d}V'}{|\boldsymbol{r} - \boldsymbol{r}'|}, \tag{2.113}$$

where, according to Figure 8.11, the retarded time is

$$t' = t - \frac{|\boldsymbol{r} - \boldsymbol{r}'|}{v}. \tag{2.114}$$

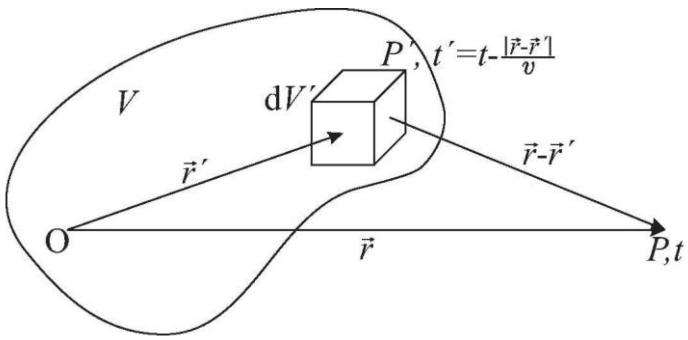

Figure 2.11: Retarded time

The retarded vector potential, which is a particularized solution of the expression (2.98), is given by an analogous formula

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{\mu}{4\pi} \int_V \frac{\boldsymbol{j}(\boldsymbol{r}', t')\mathrm{d}V'}{|\boldsymbol{r} - \boldsymbol{r}'|}. \tag{2.115}$$

The derived potentials also fulfil the Lorenz condition (2.99) due to the continuity equation. The fact that the wave equation (2.108) has two types of solution - retarded and advanced ones - is a consequence of its invariance to the inversion of time $t \rightarrow -t$. However, the boundary conditions of the problem under discussion do not possess this invariance: the source induces the electromagnetic field which propagates as a diverging wave and, as we will see, irretrievably carries the energy away. The advanced solution contradicts the relativistic principle of causality, since it inverses the causal relation between the cause (source P') and the consequence (an electromagnetic signal, received by an observer P at time $t > t'$).

2.5 Problems

1 *Lines of force of electric intensity*. Start from the differential equation of the force lines $\boldsymbol{E} \times \mathrm{d}\boldsymbol{r} = 0$ and derive an implicit equation for a single-parameter system of force lines of the field generated by a system of point charges lying on the x-axis. In the special case when the system consists only of two charges $e_1 = 2$, $x_1 = -a$ and $e_2 = -1$, $x_2 = a$, formulate the equation of the line of force that originates from the charge e_1 at an angle α with an x-axis. At what angle β does this line of force enter the charge e_2? Specify the minimum and maximum value, and such that the lines of force still end up in the charge e_2. What is the angle α of the infinite line of force originating from the charge e_1 and directed perpendicularly to the x-axis? Formulate the equation of its asymptote. *Instruction:* Integrating factor of the differential equation of the lines of force $E_y\mathrm{d}x - E_x\mathrm{d}y = 0$ for charges e_i in points x_i of the x-axis is $\mu\,(x, y) = y$.

$$[\ \sum_i e_i \cos\vartheta_i = C, \text{ kde } \cos\vartheta_i = \frac{x-x_i}{\sqrt{(x-x_i)^2+y^2}}]$$

2 *Refraction of force lines of electric intensity*. Determine the law of refraction of force lines of electric intensity $\boldsymbol{E}$ and the surface density η_P of a charge bound on the boundary of two soft dielectric mediums ε, μ. *Instructions:* $E_{1t} - E_{2t} = 0$, $\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = 0$

$$[\ \frac{\mathrm{tg}\vartheta_1}{\mathrm{tg}\vartheta_2} = \frac{E_{1t}/E_{1n}}{E_{2t}/E_{2n}} = \frac{\varepsilon_1}{\varepsilon_2};\ \eta_P = E_{1n} - E_{2n} = \frac{\varepsilon_2-\varepsilon_1}{\varepsilon_2}E_{1n}]$$

3 *Linear polarization*. What is a complex notation of a monochromatic plane wave propagating in the soft medium ε, μ in the direction of the positive axis z? The wave is linearly polarized in the direction of the x-axis.

$$[\ \widehat{E} = (E_0 e^{i(kz-\omega t)}, 0, 0),\ \ \widehat{B} = (0, \tfrac{E_0}{v}e^{i(kz-\omega t)}, 0)]$$

4 *Circular polarization*. Write the complex expression for a monochromatic plane wave propagating in a dispersed medium with the refractive index n in the direction of the negative x-axis. The wave is circularly polarized in the right-hand direction.

$$[\ \widehat{E} = (0, E_0\exp[-i\omega(\tfrac{n}{c}x+t)], E_0\exp[-i\omega(\tfrac{n}{c}x+t) - \tfrac{\pi}{2}]),$$
$$\widehat{B} = (0, -n\tfrac{E_0}{c}\exp[-i\omega(\tfrac{n}{c}x+t) - \tfrac{\pi}{2}], n\tfrac{E_0}{c}\exp[-i\omega(\tfrac{n}{c}x+t)]])]$$

5 *Vector potential of an electromagnetic wave* A monochromatic plane wave propagates in a dispersed medium in the direction of a positive axis z. Calculate its vector potential in Coulomb gauge if the wave is a) linearly polarized, b) circularly polarized.

$$[\text{ a) } \widehat{A} = \frac{E_0}{\omega}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t-\frac{\pi}{2})};\text{ b) } \boldsymbol{A} = (\tfrac{E_0}{\omega}\sin(kz-\omega t), \pm\tfrac{E_0}{\omega}\cos(kz-\omega t), 0)\ ]$$

6 cos($\boldsymbol{k}\cdot\boldsymbol{r} - \omega t$). Show that magnetic field $\boldsymbol{B}$ is automatically transverse, while transverse $\boldsymbol{E}$ requires that $b = \omega\boldsymbol{k}\cdot\boldsymbol{a}/k^2$. Show that under this condition the vector $\boldsymbol{B}$ will be perpendicular to $\boldsymbol{E}$.

7 Determine the calibration transformation that transforms the potentials from the example 8.6 ($b = \omega\boldsymbol{k}\cdot\boldsymbol{a}/k^2$) to the form $\boldsymbol{A}' = \boldsymbol{a}'\cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)$, $\varphi = 0$ corresponding to the Coulomb gauge where $\boldsymbol{k}\cdot\boldsymbol{a}' = 0$.

$$[\Lambda = -\frac{\boldsymbol{k}\cdot\boldsymbol{a}}{k^2}\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t),\ \boldsymbol{a}' = \boldsymbol{a} - (\boldsymbol{k}\cdot\boldsymbol{a})\boldsymbol{k}/k^2]$$

8 ansformation of the field from the example 8.7 with the function $\Lambda' = \mathbf{r}\cdot\mathbf{A}'$ and show that new potentials are $\boldsymbol{A}'' = (\boldsymbol{r}\cdot\boldsymbol{a}')\boldsymbol{k}\sin(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)$, $\varphi'' = -\boldsymbol{r}\cdot\boldsymbol{E}(t)$. Note that the field $\boldsymbol{A}''$ is longitudinal and φ'' is gauge-invariant; although φ'' looks like an electrostatic field, it generally depends on time. The order of magnitude $\boldsymbol{A}''$ differs from $\boldsymbol{A}'$ by the factor kr; show that in the case of light incident on atom, $kr \approx 10^{-5}$. If the electromagnetic wave falls is incident on an atom, the whole interaction in Hamilton's function can be expressed with good accuracy only by the potential $U = e\varphi = -e\boldsymbol{r}\cdot\boldsymbol{E}$ (for each electron) and the influence of the magnetic field can be neglected.

9 Show that vector potential $\boldsymbol{A}''$ from the example 8.8 leads to the same magnetic field as the original $\boldsymbol{A}$ form the example 8.6. Explain how is it possible that the vector potential $\boldsymbol{A}''$, which in many described cases is much smaller than $\boldsymbol{A}$, describes the same field?

10 Show that the time mean value of Poynting vector ($T = 2\pi/\omega$) can be calculated in the complex notation by the formula $\langle\boldsymbol{S}\rangle = \frac{1}{2}\mathrm{Re}\,(\hat{\boldsymbol{E}} \times \hat{\boldsymbol{H}}^*)$, where the asterisk denotes complex conjugate vector. Use $\mathrm{Re}\,\hat{a} = (\hat{a} + \hat{a}^*)/2$.

11 *Electromagnetic wave on the boundary* A monochromatic plane wave is incident from vacuum to the surface of a homogeneous dielectric medium ($\varepsilon \neq \varepsilon_0$, $\mu = \mu_0$) at an angle ϑ_1. Its direction of polarization forms an angle Θ with the plane of incidence.

Calculate reflectivity $R = I_1/I_0$ and permeability $T = I_2/I_0$.

$$[\,\mathcal{R} = \mathcal{R}^{\|}\cos^2\Theta + \mathcal{R}^{\perp}\sin^2\Theta,$$
$$\mathcal{T} = \mathcal{T}^{\|}\cos^2\Theta + \mathcal{T}^{\perp}\sin^2\Theta, \quad \mathcal{T}^{\|} = \sqrt{\varepsilon}(T_E^{\|})^2, \quad \mathcal{T}^{\perp} = \sqrt{\varepsilon}(T_E^{\perp})^2\,]$$

13 Determine the polarizationof the reflected andrefracted waves from the problem 8.11.

Instructions: $E_1^{\|} = R_E^{\|}E_0\cos\Theta,\ E_1^{\perp} = R_E^{\perp}E_0\sin\Theta,$
$E_2^{\|} = T_E^{\|}E_0\cos\Theta,\ E_2^{\perp} = T_E^{\perp}E_0\sin\Theta.$

[Reflected and refracted waves are linearly polarized in directions in directions forming angles to the plane of incidence

$\mathrm{tg}\Theta_1 = (R_E^{\perp}/R_E^{\|})\mathrm{tg}\Theta,\ \mathrm{tg}\Theta_2 = (T_E^{\perp}/T_E^{\|})\mathrm{tg}\Theta$]

14 Determine the solution to the problem 8.11, when the incident wave is unpolarized. To do this, average out the result of the problem 8.11 over all angles Θ.

[$\mathcal{R} = \frac{1}{2}(\mathcal{R}^{\|} + \mathcal{R}^{\perp}),\ \mathcal{T} = \frac{1}{2}(\mathcal{T}^{\|} + \mathcal{T}^{\perp})$]

2.14 Based on the result of the problem 8.11, show that the law of conservation of energy is fulfilled when the electromagnetic wave reflects and passes through the boundary. To do this, prove that the equ $\mathcal{I}_0\cos\vartheta_1 = \mathcal{I}_1\cos\vartheta_1 + \mathcal{I}_2\cos\vartheta_2.$ is valid.

15 Calculate the pressure acting on the surface of a dielectric medium by the electromagnetic wave from the problem 8.11.

[$p = w_0(\cos\vartheta_1 + \mathcal{R}\cos\vartheta_1 - \frac{\mathcal{T}}{n}\cos\vartheta_2),\ w_0 = \mathcal{I}_0/c$]