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TEACHING MATERIALS

Relativistic Physics and Astrophysics II
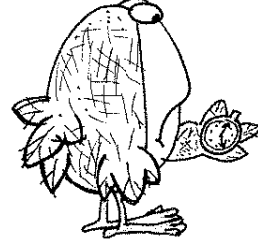
Jan Schee, Petr Slaný and Filip Blashke

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Vectors, tensors, metric

Albert Einstein made a revolutionary step when he considered to treat gravity in pure geometrical notions. He changes the notion of gravitational force with the notion of curved geometry. Therefore understanding properties of the geometric objects like vectors, tensors and metric is crucial [1, 3, 9].

E1 Contravariant components of a vector $\vec{A}$ in the coordinate system x^i are given as

$$\vec{A} = (A^1, A^2) = \left(2x^2, \frac{1}{2}x^1\right). \tag{1.1}$$

Determine the components of $\vec{A}$ with respect to coordinates $x^{i'}$, which are given via change of coordinates

$$x^{1'} = x^1 \sin x^2, \tag{1.2}$$
$$x^{2'} = x^1 \cos x^2. \tag{1.3}$$

▶ **Solution:** The Jacobi matrix reads

$$\left[\frac{\partial x^{i'}}{\partial x^j}\right] = \left[\begin{array}{cc} \frac{\partial x^{1'}}{\partial x^1} & \frac{\partial x^{1'}}{\partial x^2} \\ \frac{\partial x^{2'}}{\partial x^1} & \frac{\partial x^{2'}}{\partial x^2} \end{array}\right] = \left[\begin{array}{cc} \sin x^2 & x^1 \cos x^2 \\ \cos x^2 & -x^1 \sin x^2 \end{array}\right]. \tag{1.4}$$

The components of A^i and $A^{i'}$ of the vector $\vec{A}$ are related according to the formula

$$A^{i'} = \frac{\partial x^{i'}}{\partial x^j} A^j, \tag{1.5}$$

which gives us:

$$A^{1'} = \frac{\partial x^{1'}}{\partial x^1}A^1 + \frac{\partial x^{1'}}{\partial x^2}A^2 = 2x^2 \sin x^2 + \frac{1}{2}(x^1)^2 \cos x^2 \qquad (1.6)$$

and similarly

$$A^{2'} = \frac{\partial x^{2'}}{\partial x^1}A^1 + \frac{\partial x^{2'}}{\partial x^2}A^2 = 2x^2 \cos x^2 - \frac{1}{2}(x^1)^2 \sin x^2. \qquad (1.7)$$

E2 Let the transformation $x \to x'$ be given as

$$x^{1'} = x^1 - x^2, \qquad (1.8)$$
$$x^{2'} = x^1 + x^2. \qquad (1.9)$$

Further, let the contravariant components of the tensor $\mathbf{T}$ with respect to x be as

$$T^{11} = 1, T^{12} = T^{21} = 0 \quad \text{a} \quad T^{22} = \varrho_o. \qquad (1.10)$$

What are the contravariant components of this tensor with respect to coordinates x'?

▶ **Solution:** In this case the Jacobi matrix reads

$$\left[\frac{\partial x^{i'}}{\partial x^j}\right] = \left[\begin{array}{cc} \frac{\partial x^{1'}}{\partial x^1} & \frac{\partial x^{1'}}{\partial x^2} \\ \frac{\partial x^{2'}}{\partial x^1} & \frac{\partial x^{2'}}{\partial x^2} \end{array}\right] = \left[\begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array}\right]. \qquad (1.11)$$

Kontravariant components of the tensor T transforms according to formula

$$T^{i'j'} = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^{j'}}{\partial x^j}T^{ij}. \qquad (1.12)$$

Hence, we obtain the result

$$T^{1'1'} = T^{11} + T^{22} = 1 + \varrho_o, \qquad (1.13)$$
$$T^{1'2'} = T^{11} - T^{22} = 1 - \varrho_0, \qquad (1.14)$$
$$T^{2'1'} = T^{11} - T^{22} = 1 - \varrho_0, \qquad (1.15)$$
$$T^{2'2'} = T^{11} + T^{22} = 1 + \varrho_0. \qquad (1.16)$$

E3 Again, let us consider a change of coordinates $x \to x'$ described this time via equations

$$x^{1'} = 2x^1 + 3x^2 - x^3, \tag{1.17}$$
$$x^{2'} = -x^1 - 2x^2 + x^3, \tag{1.18}$$
$$x^{3'} = x^1. \tag{1.19}$$

Find the Jacobi matrix for this transformation and corresponding matrix for the inverse transformation.

▶ **Solution:** The Jacobi matrix is clearly as follows:

$$\left[\frac{\partial x^{i'}}{\partial x^j}\right] = \begin{bmatrix} 2 & 3 & -1 \\ -1 & -2 & 1 \\ 1 & 0 & 0 \end{bmatrix} \tag{1.20}$$

Using Gaussian elimination method we determine the inverse matrix:

$$\left[\frac{\partial x^i}{\partial x'^j}\right] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 3 & -1 \end{bmatrix}. \tag{1.21}$$

E4 The relation between Boyer-Lindquist coordinates and Kerr 'ingoing' coordinates are given as

$$\mathbf{e}_{t'} = \mathbf{e}_t, \tag{1.22}$$
$$\mathbf{e}_{u'} = -r^2\mathbf{e}_r + r^2\frac{r^2 + a^2}{\Delta}(\mathbf{e}_t + \Omega\mathbf{e}_\varphi), \tag{1.23}$$
$$\mathbf{e}_{m'} = -\frac{1}{\sin\theta}\mathbf{e}_\theta, \tag{1.24}$$
$$\mathbf{e}_{\varphi'} = \mathbf{e}_\varphi. \tag{1.25}$$

Find the Jacobi matrix for the inverse transformation.

▶ **Solution:** Comparing a general transformation relationship between the base vectors

$$\mathbf{e}_{i'} = \frac{\partial x^i}{\partial x^{i'}}\mathbf{e}_i \tag{1.26}$$

with the transformation (1.22)-(1.25) we obtain a Jacobi matrix in the form

$$\left[\frac{\partial x^a}{\partial x^{b'}}\right] = \begin{bmatrix} 1 & \frac{r^2(r^2+a^2)}{\Delta} & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sin^2\theta} & 0 \\ 0 & \frac{r^2(r^2+a^2)}{\Delta}\Omega & 0 & 1 \end{bmatrix} \tag{1.27}$$

Employing Gaussian elimination method we obtain the Jacobi matrix for the inverse transform as

$$\left[\frac{\partial x^{b'}}{\partial x^a}\right] = \begin{bmatrix} 1 & \frac{1+a^2 u'^2}{\Delta'} & 0 & 0 \\ 0 & -u'^2 & 0 & 0 \\ 0 & 0 & -\sqrt{1-m'^2} & 0 \\ 0 & \frac{au'^2}{\Delta'} & 0 & 1 \end{bmatrix}. \tag{1.28}$$

E5 Derive symmetric and anti-symmetric parts of a tensor T_{ab} which in two-dimensional space has components

$$T_{11} = 1, \quad T_{12} = \varrho, \quad T_{21} = -2\varrho \quad \text{a} \quad T_{22} = p. \tag{1.29}$$

▶ **Solution:** The symmetric part is given as

$$T_{(ab)} = \frac{1}{2}\left(T_{ab} + T_{ba}\right). \tag{1.30}$$

Which yields the result

$$T_{(11)} = 1, \quad T_{(12)} = T_{(21)} = -\frac{1}{2}\varrho \quad \text{a} \quad T_{(22)} = p. \tag{1.31}$$

On the other hand, the anti-symmetric part reads

$$T_{[ab]} = \frac{1}{2}\left(T_{ab} - T_{ba}\right), \tag{1.32}$$

which gives us

$$T_{[11]} = 0, \quad T_{(12)} = -T_{(21)} = \frac{3}{2}\varrho \quad \text{a} \quad T_{(22)} = 0. \tag{1.33}$$

E6 The line element on a 2-sphere is given as

$$\mathrm{d}l^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2. \tag{1.34}$$

At the point $P = (\theta_0, \varphi_0)$ a vector has contravariant components in the corresponding tangent space with respect to coordinate basis as follows: $V = (V^\theta, V^\varphi) = (2\varphi_0, -\theta_0)$. Determine its covariant components V_i at the point P.

▶ **Solution:** The relationship between contravariant and covariant components is

$$V_i = g_{ij}V^j, \tag{1.35}$$

where the components of the metric tensor g_{ij} are

$$[g_{ij}] = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix}. \tag{1.36}$$

Hence, at point P we have

$$V_\theta = g_{\theta j}V^j = g_{\theta\theta}V^\theta + g_{\theta\varphi}V^\varphi = 2\varphi_0 \tag{1.37}$$

and

$$V_\varphi = g_{\varphi j}V^j = g_{\varphi\theta}V^\theta + g_{\varphi\varphi}V^\varphi = -\theta_0\sin^2\theta_0. \tag{1.38}$$

E7 Prove that

(a) two-dimensional metric space with a metric

$$ds^2 = -v^2 du^2 + dv^2 \tag{1.39}$$

is a flat 2-D Minkowski spacetime with a line element

$$ds^2 = -dt^2 + dx^2. \tag{1.40}$$

(b) for unaccelerated particle the covariant component of a 4-momentum p_u is constant but p_v is not.

▶ **Solution:**

(a) It is easy to see that the transformation $(u, v) \to (t, x)$ given by

$$t = v\sinh u, \tag{1.41}$$
$$x = v\cosh u \tag{1.42}$$

transform (1.40) into (1.39). Infinitesimal increments dx and dt are given as

$$dt = dv\sinh u + v\cosh u\, du, \tag{1.43}$$
$$dx = dv\cosh u + v\sinh u\, du. \tag{1.44}$$

Substituting this into (1.40) yields the result

$$ds^2 = -dt^2 + dx^2 = \cdots = -v^2 du^2 + dv^2. \tag{1.45}$$

(b) The Hamiltonian which leads to geodesic equations has a form

$$H = \frac{1}{2} g^{ab} p_a p_b = \frac{1}{2} (p_v)^2 - \frac{1}{2} v^{-2} (p_u)^2. \tag{1.46}$$

From Hamilton equations we see

$$\frac{dp_u}{d\lambda} = -\frac{\partial H}{\partial u} = 0 \quad \Rightarrow \quad p_u = \text{const}, \tag{1.47}$$

$$\frac{dp_v}{d\lambda} = -\frac{\partial H}{\partial v} = -v^{-3}(p_u)^2 \quad \Rightarrow \quad p_v \neq \text{const}. \tag{1.48}$$

E8 Prove that the conformal transformation of the metric, i.e. $g_{ab} \to f(x^c) g_{ab}$ for an arbitrary function f, conserves angles. Show that all null curves remain null after the transformation.

▶ **Solution:** At a given point, local angle α between two vectors A, B is given by a relation

$$\cos \alpha = \frac{A \cdot B}{|A||B|} = \frac{g_{ab} A^a B^b}{\sqrt{g_{ab} A^a A^b} \sqrt{g_{ab} B^a B^b}}. \tag{1.49}$$

Calculating the angle α' between A, B with respect to metric $f g_{ab}$, we obtain

$$\cos \alpha' = \frac{f g_{ab} A^a B^b}{\sqrt{f g_{ab} A^a A^b} \sqrt{f g_{ab} B^a B^b}} = \frac{f}{f} \frac{g_{ab} A^a B^b}{\sqrt{g_{ab} A^a A^b} \sqrt{g_{ab} B^a B^b}} = \cos \alpha. \tag{1.50}$$

Further, let k be a null vector, that is a vector which components satisfy

$$g_{ab} k^a k^b = 0. \tag{1.51}$$

In conformal metric, however

$$f g_{ab} k^a k^b = 0. \tag{1.52}$$

Hence the null vectors remains null after a conformal transformation.

E9 Determine a magnitude of 4-velocity and 4-momentum with contravariant components

$$U^i = \frac{dx^i}{d\tau} \quad \text{a} \quad P^i = m U^i, \tag{1.53}$$

where m is the rest mass of a particle and τ is its proper time.

▶ **Solution:** In a metric space, a square of the vector $\mathbf{U} = U^i \mathbf{e}_i$ reads

$$U^2 = \mathbf{U} \cdot \mathbf{U} = g_{ij} U^i U^j = g_{ij} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau}. \tag{1.54}$$

However, at the same time

$$\mathrm{d}s^2 = g_{ij}\mathrm{d}x^i\mathrm{d}x^j \quad \Rightarrow \quad \frac{\mathrm{d}s^2}{\mathrm{d}\tau^2} = g_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}\tau}\frac{\mathrm{d}x^j}{\mathrm{d}\tau}. \tag{1.55}$$

The relationship between spacetime interval $\mathrm{d}s$ and proper time $\mathrm{d}\tau$ is (with the speed of light $c = 1$)
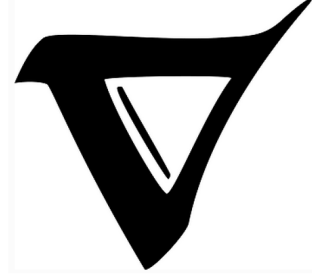
$$\mathrm{d}s^2 = -\mathrm{d}\tau^2. \tag{1.56}$$

These equations points to the result

$$U^2 = g_{ij}U^iU^j = -1 \quad \mathrm{a} \quad P^2 = g_{ij}P^iP^j = -m^2. \tag{1.57}$$

2

Covariant derivative, connection, curvature

In order to describe change of a vector or a tensor in curved spacetime we need a new operation of derivative. Here, the properties of covariant derivative, connection and curvature are practised. For detailed definitions see [1, 3, 9].

E1 Determine transformation properties of a geometrical object with components $\partial V^i / \partial x^j$

▶ **Solution:** Let us consider a transformation $x' \to x$ together with its inverse $x \to x'$. We calculate

$$
\begin{aligned}
\frac{\partial V^{i'}}{\partial x^{j'}} &= \frac{\partial x^i}{\partial x^{j'}} \frac{\partial V^{i'}}{\partial x^i} = \frac{\partial x^i}{\partial x^{j'}} \frac{\partial}{\partial x^i} \left(\frac{\partial x^{i'}}{\partial x^j} V^j \right) \\
&= \frac{\partial x^i}{\partial x^{j'}} \left(\frac{\partial x^{i'}}{\partial x^j} \frac{\partial V^j}{\partial x^i} + V^j \frac{\partial^2 x^{i'}}{\partial x^i \partial x^j} \right) \\
&= \frac{\partial x^i}{\partial x^{j'}} \frac{\partial x^{i'}}{\partial x^j} \frac{\partial V^j}{\partial x^i} + \frac{\partial x^i}{\partial x^{j'}} \frac{\partial^2 x^{i'}}{\partial x^i \partial x^j} V^j .
\end{aligned}
\tag{2.1}
$$

First term of the resulting transformation relation is a transformation of a (1,1) tensor but the non-zero second term causes that the partial derivatives of vector components do not behave as a tensor.

E2 From the definition of covariant derivative determine a transformation rule for components of affine connection Γ^i_{jk}.

▶ **Solution:** Covariant derivative of contravariant components of a vector **V** is

$$\nabla_i V^j = \partial_i V^j + \Gamma^j{}_{ki} V^k. \tag{2.2}$$

Considering a change of coordinates $x' \to x$ we have

$$
\begin{aligned}
\nabla_{i'} V^{j'} &= \partial_{i'} V^{j'} + \Gamma^{j'}{}_{k'i'} V^{k'} \\
&= \frac{\partial x^{j'}}{\partial x^j} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial V^j}{\partial x^i} + \frac{\partial x^i}{\partial x^{i'}} \frac{\partial^2 x^{j'}}{\partial x^i \partial x^j} V^j + \frac{\partial x^{k'}}{\partial x^k} \Gamma^{j'}{}_{k'i'} V^k \\
&= \text{|from def. (2.2)|} = \frac{\partial x^{j'}}{\partial x^j} \frac{\partial x^i}{\partial x^{i'}} \left(\frac{\partial V^j}{\partial x^i} + \Gamma^j{}_{ki} V^k \right). \tag{2.3}
\end{aligned}
$$

Hence, we get

$$\frac{\partial x^{j'}}{\partial x^j} \frac{\partial x^i}{\partial x^{i'}} \Gamma^j{}_{ki} V^k = \frac{\partial x^i}{\partial x^{i'}} \frac{\partial^2 x^{j'}}{\partial x^i \partial x^j} V^j + \frac{\partial x^{k'}}{\partial x^k} \Gamma^{j'}{}_{k'i'} V^k, \tag{2.4}$$

which after some algebra yields a result

$$\Gamma^{j'}{}_{k'i'} = \frac{\partial x^k}{\partial x^{k'}} \frac{\partial x^{j'}}{\partial x^j} \frac{\partial x^i}{\partial x^{i'}} \Gamma^j{}_{ki} - \frac{\partial x^k}{\partial x^{k'}} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial^2 x^{j'}}{\partial x^i \partial x^k}. \tag{2.5}$$

It is clear that the components of affine connection do not transform as a tensor.

E3 Determine how the components of an affine connection transform.

▶ **Solution:** Let us consider a free falling observer with coordinates ξ^α. A free particle with respect to this observer will obey equations of motion

$$\frac{\mathrm{d}^2 \xi^\alpha}{\mathrm{d}\tau^2} = 0. \tag{2.6}$$

From the point of view of an observer on the surface of a star (for example) with coordinates x^α the equations of motion will be as follows. Since the coordinates ξ^α are functions of x^α from (2.6) we get

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau} \left(\frac{\mathrm{d}\xi^\alpha}{\mathrm{d}\tau} \right) &= \frac{\mathrm{d}}{\mathrm{d}\tau} \left(\frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \right) \\
&= \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \frac{\partial^2 \xi^\alpha}{\partial x^\nu \partial x^\mu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = 0 \tag{2.7}
\end{aligned}
$$

We multiply this equation by $\partial x^\sigma / \partial \xi^\alpha$ to obtain

$$\frac{\mathrm{d}^2 x^\sigma}{\mathrm{d}\tau^2} + \Gamma^\sigma{}_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}, \tag{2.8}$$

where we introduced

$$\Gamma^{\sigma}{}_{\mu\nu} = \frac{\partial x^{\sigma}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial x^{\mu} \partial x^{\nu}}. \tag{2.9}$$

Further, let us consider a transformation of coordinates $x'^{\mu} = x'^{\mu}(x^{\nu})$ and from (2.9) we determine how the affine connection transforms.

First, let us establish following relations

$$\frac{\partial x^{\mu'}}{\partial \xi^i} = \frac{\partial x^{\mu'}(x^{\sigma}(\xi^j))}{\partial \xi^i} = \frac{\partial x^{\mu'}}{\partial x^{\sigma}} \frac{\partial x^{\sigma}}{\partial \xi^i} \tag{2.10}$$

and

$$\frac{\partial \xi^i}{\partial x^{\beta'}} = \frac{\partial \xi^i(x^{\alpha}(x^{\alpha'}))}{\partial x^{\beta'}} = \frac{\partial \xi^i}{\partial x^{\beta}} \frac{\partial x^{\beta}}{\partial x^{\beta'}}. \tag{2.11}$$

Using these we calculate:

$$
\begin{aligned}
\Gamma^{\alpha'}{}_{\beta'\gamma'} &= \frac{\partial x^{\alpha'}}{\partial \xi^{\sigma}} \frac{\partial^2 \xi^{\sigma}}{\partial x^{\beta'} \partial x^{\gamma'}} = \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \frac{\partial x^{\alpha}}{\partial \xi^{\sigma}} \frac{\partial}{\partial x^{\beta'}} \left(\frac{\partial x^{\gamma}}{\partial x^{\gamma'}} \frac{\partial \xi^{\sigma}}{\partial x^{\gamma}} \right) \\
&= \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \frac{\partial x^{\alpha}}{\partial \xi^{\sigma}} \left(\frac{\partial \xi^{\sigma}}{\partial x^{\gamma}} \frac{\partial^2 x^{\gamma}}{\partial x^{\gamma'} \partial x^{\beta'}} + \frac{\partial x^{\gamma}}{\partial x^{\gamma'}} \frac{\partial x^{\beta}}{\partial x^{\beta'}} \frac{\partial^2 \xi^{\sigma}}{\partial x^{\beta} \partial x^{\gamma}} \right) \\
&= \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \frac{\partial x^{\gamma}}{\partial x^{\gamma'}} \frac{\partial x^{\beta}}{\partial x^{\beta'}} \Gamma^{\alpha}{}_{\beta\gamma} + \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \frac{\partial^2 x^{\alpha}}{\partial x^{\beta'} \partial x^{\gamma'}}. \tag{2.12}
\end{aligned}
$$

We obtained and alternative form of a transformation rule for components of an affine connection (compare with (2.5)).

E4 Show that the quantity

$$A_{\mu\nu} = \frac{\partial A_{\mu}}{\partial x^{\nu}} - \frac{\partial A_{\nu}}{\partial x^{\mu}} \tag{2.13}$$

transforms as a tensor.

▶ **Solution:** Partial derivatives of covariant components of a vector transform as

$$
\begin{aligned}
\frac{\partial A_{\alpha'}}{\partial x^{\beta'}} &= \frac{\partial x^{\beta}}{\partial x^{\beta'}} \frac{\partial A_{\alpha'}}{\partial x^{\beta}} = \frac{\partial x^{\beta}}{\partial x^{\beta'}} \frac{\partial}{\partial x^{\beta}} \left(\frac{\partial x^{\alpha}}{\partial x^{\alpha'}} A_{\alpha} \right) \\
&= \frac{\partial x^{\alpha}}{\partial x^{\alpha'}} \frac{\partial x^{\beta}}{\partial x^{\beta'}} \frac{\partial A_{\alpha}}{\partial x^{\beta}} + \frac{\partial^2 x^{\alpha}}{\partial x^{\alpha'} \partial x^{\beta'}} A_{\alpha}. \tag{2.14}
\end{aligned}
$$

Now it is a simple matter of putting this expression into (2.13) to obtain:

$$
\begin{aligned}
A_{\alpha'\beta'} &= \frac{\partial A_{\alpha'}}{\partial x^{\beta'}} - \frac{\partial A_{\beta'}}{\partial x^{\alpha'}} \\
&= \frac{\partial x^{\alpha}}{\partial x^{\alpha'}} \frac{\partial x^{\beta}}{\partial x^{\beta'}} \frac{\partial A_{\alpha}}{\partial x^{\beta}} + \frac{\partial^2 x^{\sigma}}{\partial x^{\alpha'}\partial x^{\beta'}} A_{\sigma} \\
&\quad - \frac{\partial x^{\beta}}{\partial x^{\beta'}} \frac{\partial x^{\alpha}}{\partial x^{\alpha'}} \frac{\partial A_{\beta}}{\partial x^{\alpha}} - \frac{\partial^2 x^{\sigma}}{\partial x^{\beta'}\partial x^{\alpha'}} A_{\sigma} \\
&= \frac{\partial x^{\alpha}}{\partial x^{\alpha'}} \frac{\partial x^{\beta}}{\partial x^{\beta'}} \left(\frac{\partial A_{\alpha}}{\partial x^{\beta}} - \frac{\partial A_{\beta}}{\partial x^{\alpha}} \right) = \frac{\partial x^{\alpha}}{\partial x^{\alpha'}} \frac{\partial x^{\beta}}{\partial x^{\beta'}} A_{\alpha\beta}. \quad (2.15)
\end{aligned}
$$

As we see, the quantity defined in (2.13) indeed transforms as a tensor!

E5 Determine components of a metric connection on a 2-sphere, which has a line element

$$
\mathrm{d}l^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2. \quad (2.16)
$$

▶ **Solution:** Components of a metric connection in coordinate basis are given by a formula

$$
\Gamma^i{}_{jk} = \frac{1}{2} g^{is}(-g_{jk,s} + g_{sj,k} + g_{ks,j}) \quad (2.17)
$$

from which we easily obtain all non-zero components as

$$
\Gamma^\theta{}_{\varphi\varphi} = -\cos\theta \sin\theta, \quad \Gamma^\varphi{}_{\theta\varphi} = \Gamma^\varphi{}_{\varphi\theta} = \cot g\,\theta. \quad (2.18)
$$

E6 Calculate elements of a Riemann tensor, a Ricci tensor and a Ricci scalar given the following line elements:

(a)
$$
\mathrm{d}l^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\varphi^2, \quad (2.19)
$$

(b)
$$
\mathrm{d}l^2 = \mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi^2. \quad (2.20)
$$

▶ **Solution:** The components of a Riemann tensor are by definition

$$
R^i{}_{jkl} \equiv \Gamma^i{}_{jl,k} - \Gamma^i{}_{jk,l} + \Gamma^s{}_{jl}\Gamma^i{}_{ks} - \Gamma^s{}_{jk}\Gamma^i{}_{ls}, \quad (2.21)
$$

while the components of a Ricci tensor and of a Ricci scalar can be obtained as

$$
R_{jl} = R^i{}_{jil}, \quad R = R^i{}_i = g^{ij} R_{ij}. \quad (2.22)
$$

In two-dimensional space there is only one independent component of a Riemann tensor. The remaining non-zero components follows from its symmetries:

$$R_{ijkl} = -R_{ijlk} = -R_{jikl} = R_{klij}, \tag{2.23}$$
$$R^i{}_{jkl} + R^i{}_{ljk} + R^i{}_{klj} \equiv 0. \tag{2.24}$$

By direct calculation we get

(a)
$$R^i{}_{jkl} = 0, \quad R_{ik} = 0 \quad \text{and} \quad R = 0 \quad \text{for all } i, j, k, l. \tag{2.25}$$

Thus, we get a flat space.

(b) The independent component is, e.g.

$$R^\theta{}_{\varphi\theta\varphi} = \sin^2\theta. \tag{2.26}$$
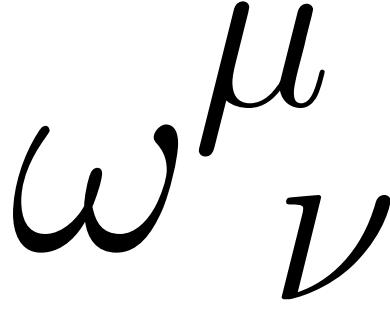
Hence,

$$R_{\varphi\varphi} = R^i{}_{\varphi i\varphi} = R^\theta{}_{\varphi\theta\varphi} + R^\varphi{}_{\varphi\varphi\varphi} = \sin^2\theta, \tag{2.27}$$
$$R_{\theta\theta} = R^i{}_{\theta i\theta} = R^\theta{}_{\theta\theta\theta} + R^\varphi{}_{\theta\varphi\theta} = g^{\varphi\varphi}g_{\theta\theta}R^\theta{}_{\varphi\theta\varphi} =$$
$$\frac{1}{\sin^2\theta}\sin^2\theta = 1 \tag{2.28}$$

and

$$R = g^{lj}R_{lj} = g^{\varphi\varphi}R_{\varphi\varphi} + g^{\theta\theta}R_{\theta\theta} = \frac{1}{\sin^2\theta}\sin^2\theta + 1 = 2. \tag{2.29}$$

$$\omega^\mu{}_\nu$$

3

Differential geometry, Maurer-Cartan structure equations

 The concept of differental forms enables one a coordinate independent definition of geometric object. Introduction of tetrades simplifies analysis of physical phenomena in strong gravity since there physical laws take, locally, the same form as in flat spacetime [1, 3, 9, 7].

Let us consider a tetrade of base vector e_a, where $a = (0,\,1,\,2,\,3)$, which does not need to be orthonormal. A dual tetrad of base 1-forms ω^a is defined via relation

$$e_a \cdot \omega^b = \delta^a_b, \tag{3.1}$$

where δ^a_b is a four-dimensional Kronecker's delta. The spacetime interval is then given as

$$\mathrm{d}s^2 = g_{ab}\omega^a\omega^b, \tag{3.2}$$

where

$$g_{ab} = e_a \cdot e_b = g_{\alpha\beta}e^\alpha_a e^\beta_b \tag{3.3}$$

are tetrad components of the metric. Usually, we choose an orthonormal basis so that $g_{ab} = \eta_{ab} = \mathrm{diag}(-1,\,1,\,1,\,1)$.

In curved spacetimes an important role is played by the spin connection $\Omega^a{}_b$, which are connected to tetrad components of Christoffel's symbols Γ^a_{bc} via

relationship

$$\Omega^a{}_b = \Gamma^a_{bc}\omega^c. \tag{3.4}$$

The exterior derivative of base vectors is defined via spin connection:

$$de_a = e_b \otimes \Omega^b{}_a. \tag{3.5}$$

Then, exterior derivative of the metric satisfies

$$dg_{ab} = \Omega_{ab} + \Omega_{ba}. \tag{3.6}$$

For the spin connection there an important *1. Maurer-Cartan structure equation*, which in the case of torsion-free spacetime (with metric connection) has a form

$$d\omega^a = -\Omega^a{}_b \wedge \omega^b, \tag{3.7}$$

where $\wedge$ indicates exterior product of 1-forms.

To describe a curved spacetime is useful to introduce 2-forms of curvative, which are related with tetrad components of Riemann's tensor via relation

$$\mathcal{R}^a{}_b = \frac{1}{2} R^a{}_{bcd}\,\omega^c \wedge \omega^d. \tag{3.8}$$

These 2-forms then satisfy *2. Maurer Cartan structure equation*

$$\mathcal{R}^a{}_b = d\Omega^a{}_b + \Omega^a{}_c \wedge \Omega^c{}_b. \tag{3.9}$$

E1 Using Maurer-Cartan structure equations determine a spin connection, 2-forms of curvature and based on those calculate tetrad components of Einstein tensor for Friedmann closed universe.

▶ **Solution:** Spacetime interval between two close events in Friedmann closed universe has a form

$$ds^2 = -dt^2 + a(t)^2[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\varphi^2)], \tag{3.10}$$

where (χ, θ, φ) are co-moving space coordinates and t is a cosmic time as perceived by a typical observer carried away by cosmic expansion.

Orthonormal tetrad of base 1-form of such an observer is then given as

$$
\begin{align}
\omega^{\hat{t}} &= \mathrm{d}t, \tag{3.11}\\
\omega^{\hat{\chi}} &= a\mathrm{d}\chi, \tag{3.12}\\
\omega^{\hat{\theta}} &= a\sin\chi\mathrm{d}\theta, \tag{3.13}\\
\omega^{\hat{\varphi}} &= a\sin\chi\sin\theta\mathrm{d}\varphi. \tag{3.14}
\end{align}
$$

With respect to this orthonormal base ($g_{ab} = \eta_{ab}$) the spin connection is antisymmetric (viz (3.6)): $\Omega_{ab} = -\Omega_{ba}$. Then, there exist 6 independent components of a spin connection, which are determined by 1. Maurer-Cartan structure equation (3.7) using antisymmetry of exterior product and the property of the exterior derivative, namely $\mathrm{d}^2 = 0$ (Poincaré lemma: *the boundary of the boundary is zero*):

$$
\begin{align}
\Omega^{\hat{k}}{}_{\hat{t}} &= \Omega^{\hat{t}}{}_{\hat{k}} = \frac{\dot{a}}{a}\omega^{\hat{k}}, \quad k = \{\chi,\,\theta,\,\varphi\}, \tag{3.15}\\
\Omega^{\hat{\theta}}{}_{\hat{\chi}} &= -\Omega^{\hat{\chi}}{}_{\hat{\theta}} = \frac{1}{a}\mathrm{cotg}\,\chi\omega^{\hat{\theta}} = \cos\chi\mathrm{d}\theta, \tag{3.16}\\
\Omega^{\hat{\varphi}}{}_{\hat{\chi}} &= -\Omega^{\hat{\chi}}{}_{\hat{\varphi}} = \frac{1}{a}\mathrm{cotg}\,\chi\omega^{\hat{\varphi}} = \cos\chi\sin\theta\mathrm{d}\varphi, \tag{3.17}\\
\Omega^{\hat{\varphi}}{}_{\hat{\theta}} &= -\Omega^{\hat{\theta}}{}_{\hat{\varphi}} = \frac{1}{a\sin\chi}\mathrm{cotg}\,\theta\omega^{\hat{\varphi}} = \cos\theta\mathrm{d}\varphi. \tag{3.18}
\end{align}
$$

The 2-forms of curvature can be determined by putting the spin connection into 2. Maurer-Cartan structure equation (3.9). The non-zero components are

$$
\begin{align}
\mathcal{R}^{\hat{t}}{}_{\hat{k}} &= \frac{\ddot{a}}{a}\omega^{\hat{t}}\wedge\omega^{\hat{k}}, \quad k = \{\chi,\,\theta,\,\varphi\}, \tag{3.19}\\
\mathcal{R}^{\hat{k}}{}_{\hat{l}} &= \left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{a^2}\right]\omega^{\hat{k}}\wedge\omega^{\hat{l}}, \quad k,\,l = \{\chi,\,\theta,\,\varphi\},\ k \neq l. \tag{3.20}
\end{align}
$$

Non-zero components of a Riemann tensor in orthonormal tetrad can be extracted directly from the relation (3.8). We get:

$$
\begin{align}
R^{\hat{t}}{}_{\hat{\chi}\hat{t}\hat{\chi}} &= R^{\hat{t}}{}_{\hat{\theta}\hat{t}\hat{\theta}} = R^{\hat{t}}{}_{\hat{\varphi}\hat{t}\hat{\varphi}} = \frac{\ddot{a}}{a}, \tag{3.21}\\
R^{\hat{\chi}}{}_{\hat{\theta}\hat{\chi}\hat{\theta}} &= R^{\hat{\chi}}{}_{\hat{\varphi}\hat{\chi}\hat{\varphi}} = R^{\hat{\theta}}{}_{\hat{\varphi}\hat{\theta}\hat{\varphi}} = \left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{a^2}. \tag{3.22}
\end{align}
$$

Contracting Riemann tensor we obtain tetrad components of Ricci tensor and by its contraction we get Ricci scalar:

$$R_{\hat{t}\hat{t}} = R^{\hat{k}}{}_{\hat{t}\hat{k}\hat{t}} = -3\frac{\ddot{a}}{a}, \tag{3.23}$$

$$R_{\hat{\chi}\hat{\chi}} = R^{\hat{k}}{}_{\hat{\chi}\hat{k}\hat{\chi}} = \frac{\ddot{a}}{a} + 2\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{a^2}\right] = R_{\hat{\theta}\hat{\theta}} = R_{\hat{\varphi}\hat{\varphi}}, \tag{3.24}$$

$$R = R^{\hat{k}}{}_{\hat{k}} = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{a^2}\right]. \tag{3.25}$$

Finally, the components of the Einstein tensor in orthonormal tetrad are determined via $G_{\hat{a}\hat{b}} = R_{\hat{a}\hat{b}} - (1/2)\eta_{\hat{a}\hat{b}}R$:

$$G_{\hat{t}\hat{t}} = 3\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{a^2}\right], \tag{3.26}$$

$$G_{\hat{\chi}\hat{\chi}} = G_{\hat{\theta}\hat{\theta}} = G_{\hat{\varphi}\hat{\varphi}} = -\left[2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{a^2}\right]. \tag{3.27}$$

E2 Using Maurer-Cartan structure equations determine the components of Einstein tensor in Schwarzschild spacetime. Use an orthonormal basis.

▶ **Solution:** Spacetime interval between two neighbouring events in Schwarzschild spacetime has a form

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{3.28}$$

The orthonormal tetrad of base 1-forms reads:

$$\omega^{\hat{t}} = e^{\alpha}dt, \tag{3.29}$$

$$\omega^{\hat{r}} = e^{\beta}dr, \tag{3.30}$$

$$\omega^{\hat{\theta}} = rd\theta, \tag{3.31}$$

$$\omega^{\hat{\theta}} = r\sin\theta d\varphi. \tag{3.32}$$

The ensuing calculation follows the same beats as in the previous example.

The non-zero components of Riemann tensor are, then:

$$R^{\hat{t}}{}_{\hat{r}\hat{t}\hat{r}} = -e^{-2\beta}(\alpha'' + \alpha'^2 - \alpha'\beta'), \tag{3.33}$$

$$R^{\hat{t}}{}_{\hat{\theta}\hat{t}\hat{\theta}} = R^{\hat{t}}{}_{\hat{\varphi}\hat{t}\hat{\varphi}} = -\frac{1}{r}\alpha' e^{-2\beta}, \tag{3.34}$$

$$R^{\hat{r}}{}_{\hat{\theta}\hat{r}\hat{\theta}} = R^{\hat{r}}{}_{\hat{\varphi}\hat{r}\hat{\varphi}} = \frac{1}{r}\beta' e^{-2\beta}, \tag{3.35}$$

$$R^{\hat{\theta}}{}_{\hat{\varphi}\hat{\theta}\hat{\varphi}} = \frac{1}{r^2}(1 - e^{-2\beta}). \tag{3.36}$$

Using these we obtain non-zero components of the Einstein tensor as:

$$G_{\hat{t}\hat{t}} = \frac{2}{r}\beta' e^{-2\beta} + \frac{1}{r^2}(1 - e^{-2\beta}), \tag{3.37}$$

$$G_{\hat{r}\hat{r}} = \frac{2}{r}\alpha' e^{-2\beta} - \frac{1}{r^2}(1 - e^{-2\beta}), \tag{3.38}$$

$$G_{\hat{\theta}\hat{\theta}} = G_{\hat{\varphi}\hat{\varphi}} = \frac{1}{r}e^{-2\beta}(r\alpha'' + r\alpha'^2 - r\alpha'\beta' + \alpha' - \beta'). \tag{3.39}$$

E3 Let us consider an arbitrary, stationary, axially symmetric spacetime.

(a) Calculate angular velocity for ZAMO observers. (ZAMO = Zero Angular Momentum Observer)

(b) Tetrad of base 1-forms of a ZAMO observer in Boyer-Lindquist coordinates is given by

$$\omega^{\hat{t}} = |g_{tt} - \omega^2 g_{\varphi\varphi}|^{1/2} dt \tag{3.40}$$

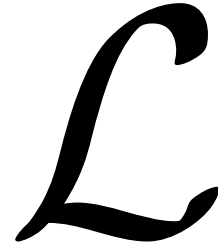$$\omega^{\hat{\varphi}} = (g_{\varphi\varphi})^{1/2}(d\varphi - \omega dt) \tag{3.41}$$

$$\omega^{\hat{r}} = (g_{rr})^{1/2} dr \tag{3.42}$$

$$\omega^{\hat{\theta}} = (g_{\theta\theta})^{1/2} d\theta. \tag{3.43}$$

Find vectors of dual base.

(c) ZAMO is not an inertial observer. Find its 4-acceleration.

4

Lie derivative, Killing vectors*

$\mathcal{L}$

Killing vectors uncover symmetries of the spacetime, they are solutions of the Killing equation. The symmetries of the spacetime are connected with constants of motion. Using the symmetries one can strongly simplify equations of motion to get qualitative and quantitative insight into particular physical problem [1, 3, 9].

E1 Show that Lie derivative of a metric is zero $\mathcal{L}_\xi g = 0$ is equivalent to satisfying Killing equation $\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0$.

E2 Show that

(a) a commutator of Killing vector fields is again a Killing vector field,

(b) a linear combination of Killing vectors with constant coefficients is a Killing vector too.

E3 Find all Killing vectors of a 2-sphere.

▶ **Solution:** A line element on a surface of a sphere can be written in angular coordinates (θ, φ) as:

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2. \tag{4.1}$$

Writting down Killing equation $\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0$ for specific values of indices

μ, ν gives:

$$\mu = \nu = \theta: \qquad \partial_\theta \xi_\theta = 0, \tag{4.2}$$

$$\mu = \nu = \varphi: \qquad \partial_\varphi \xi_\varphi = -\sin\theta\cos\theta \xi_\theta, \tag{4.3}$$

$$\mu = \theta, \ \nu = \varphi: \quad \partial_\varphi \xi_\theta + \partial_\theta \xi_\varphi = 2\mathrm{cotg}\,\theta \xi_\varphi. \tag{4.4}$$

The first equation (4.2) is solved as

$$\xi_\theta = f(\varphi), \tag{4.5}$$

where $f(\varphi)$ is as yet unknown function. Putting this into (4.3) we obtain

$$\xi_\varphi = -F(\varphi)\sin\theta\cos\theta + g(\theta), \tag{4.6}$$

where $F(\varphi) = \int f(\varphi)\mathrm{d}\varphi$ and $g(\theta)$ are again unknown functions. Substituting for ξ_θ and ξ_φ in (4.4) we get, after some algebra, an equation in separable form:

$$\frac{\mathrm{d}f}{\mathrm{d}\varphi} + F(\varphi) = -\left(\frac{\mathrm{d}g}{\mathrm{d}\theta} - 2g\,\mathrm{cotg}\,\theta\right). \tag{4.7}$$

Since the left side is a function of φ while the right side of θ, the equality is maintained only if both sides are equal to the same constant, say k. Thus, the equation (4.7) reduces to two pieces:

$$\frac{\mathrm{d}f}{\mathrm{d}\varphi} + \int f(\varphi)\mathrm{d}\varphi = k, \tag{4.8}$$

$$\frac{\mathrm{d}g}{\mathrm{d}\theta} - 2g\,\mathrm{cotg}\,\theta = -k. \tag{4.9}$$

If we differentiate (4.8) by φ the result is

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\varphi^2} + f(\varphi) = 0 \qquad \longrightarrow \qquad \begin{aligned} f(\varphi) &= a\sin\varphi + b\cos\varphi, \quad &(4.10)\\ F(\varphi) &= -a\cos\varphi + b\sin\varphi. \quad &(4.11) \end{aligned}$$

Plugging $f(\varphi)$ into (4.8) we learn that $k = 0$. Then, the equation (4.9) is easily solved by the method of separation of variables, which after integration yields:

$$g(\theta) = c\sin^2\theta. \tag{4.12}$$

Covariant components of a Killing vector are given by (4.5) and (4.6). After substituting for f, F a g we obtain the result:

$$\xi_\theta = a\sin\varphi + b\cos\varphi, \tag{4.13}$$

$$\xi_\varphi = \sin\varphi\cos\varphi(a\cos\varphi - b\sin\varphi) + c\sin^2\theta. \tag{4.14}$$

We determine contravariant components through relation $\xi^i = g^{ij}\xi_j$. Hence, the Killing vector in coordinate basis is given as

$$\xi = \xi^\theta \frac{\partial}{\partial\theta} + \xi^\varphi \frac{\partial}{\partial\varphi} =$$

$$(a\sin\varphi + b\cos\varphi)\frac{\partial}{\partial\theta} + [\text{cotg}\,\theta(a\cos\varphi - b\sin\varphi) + c]\frac{\partial}{\partial\varphi}. \ (4.15)$$

We can see that due to the existence of integration constants the Killing vector can be written as a linear combination of two basic Killing vectors

$$\mathbf{X_1} = \sin\varphi\frac{\partial}{\partial\theta} + \text{cotg}\,\theta\cos\varphi\frac{\partial}{\partial\varphi}, \qquad (4.16)$$

$$\mathbf{X_2} = \cos\varphi\frac{\partial}{\partial\theta} - \text{cotg}\,\theta\cos\varphi\frac{\partial}{\partial\varphi}, \qquad (4.17)$$

$$\mathbf{X_3} = \frac{\partial}{\partial\varphi}. \qquad (4.18)$$

These satisfy the commutation relation

$$[\mathbf{X_i},\,\mathbf{X_j}] = \epsilon_{ijk}\mathbf{X_k}, \qquad (4.19)$$

which are analogous to the well-known commutation relations for operators of angular momentum. Correspondingly, vectors $\mathbf{X_i}$ are generators of the SU(2) group.

E4 If $\xi(x)$ is a Killing vector field and u a tangent vector to a geodetic, show that $\xi \cdot u$ is a constant along this geodetic.

E5 If ξ is a Killing vector and T is a energy-momentum tensor , show that $J^\mu = T^{\mu\nu}\xi_\nu$ is a conserved quantity. What is the corresponding J in the case of a timelike Killing vector?

5
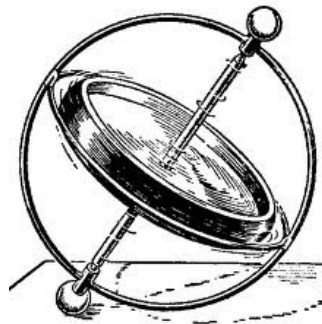
Momentum, spin*

Angular momentum and spin are key concepts describing rotation. In general relativity one needs covariant equations describing physical quantities. This section provides examples supporting students understanding of those concepts [2, 3].

$\boxed{\text{E1}}$ Show that the total angular momentum of an isolated system in Minkowski spacetime

$$J^{\alpha\beta} \equiv \int (x^\alpha T^{\beta 0} - x^\beta T^{\alpha 0})\mathrm{d}^3 x. \tag{5.1}$$

(a) is a conserved quantity,

(b) is not invariant under the translation by a constant vector a^α.

▶ **Solution:** (a) Let us split the conserved charges into spatial components J^{ij} – these are consequence of invariance under rotations and, hence, are angular momenta of the fields – and other three elements J^{i0}, which represent conserved charges resulting from invariance under boosts. Let us demonstrate that they are conserved one at the time. For angular momenta we have

$$
\begin{aligned}
\frac{\mathrm{d}J^{ij}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t} \int \left(x^i T^{j0} - x^j T^{i0}\right)\mathrm{d}^3 x = \int \left(x^i \partial_0 T^{j0} - x^j \partial_0 T^{i0}\right)\mathrm{d}^3 x \\
&= -\int \left(x^i \partial_k T^{jk} - x^j \partial_k T^{ik}\right)\mathrm{d}^3 x = \int \left(T^{ji} - T^{ij}\right)\mathrm{d}^3 x = 0. \quad (5.2)
\end{aligned}
$$

In the second line, we have first used equation of continuity $\partial_\mu T^{i\mu} = \partial_0 T^{i0} + \partial_k T^{ik} = 0$ and then we used per partes to move derivatives onto

the x^i's. Finally, the conservation of the angular momenta follows from symmetry of energy-momentum tensor $T^{ij} = T^{ji}$. The conservation of 'boost charges' is demonstrated as follows:

$$
\begin{aligned}
\frac{\mathrm{d}J^{i0}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t} \int \left(x^i T^{00} - t\, T^{i0} \right) \mathrm{d}^3 x = \int \left(x^i \partial_0 T^{00} - T^{i0} - t \partial_0 T^{i0} \right) \mathrm{d}^3 x \\
&= \int \left(-x^i \partial_k T^{0k} - T^{i0} + t \partial_k T^{ik} \right) \mathrm{d}^3 x = \int \left(T^{0i} - T^{i0} + \partial_k (t\, T^{ik}) \right) \mathrm{d}^3 x \\
&= 0.
\end{aligned} \tag{5.3}
$$

Again, in the second line we have used the fact that $\partial_\mu T^{0\mu} = 0$ and then used per partes to move ∂_k onto x^i. Then, using the fact that $T^{0i} = T^{i0}$ and further noticing that the last term at the end of the second line is a boundary term which we assume to vanish, gives us the desired result.

(b) Due to the explicit dependence of $J^{\alpha\beta}$ on coordinates in the argument of the integral a shift $x^\alpha \to x^\alpha + a^\alpha$ translates the tensor as

$$
J^{\alpha\beta} \to J^{\alpha\beta} + \int \left(a^\alpha T^{\beta 0} - a^\beta T^{\alpha 0} \right) \mathrm{d}^3 x = J^{\alpha\beta} + a^\alpha P^\beta - a^\beta P^\alpha, \tag{5.4}
$$

where P^α is the total momentum. Obviously, the quantity $a^\alpha P^\beta - a^\beta P^\alpha$ does not vanish in general and, therefore, $J^{\alpha\beta}$ is not invariant under constant translation of coordinates.

E2 Show that a spin 4-vector in flat spacetime

$$
S_\alpha \equiv -\frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} J^{\beta\gamma} u^\delta \tag{5.5}
$$

is

(a) a conserved quantity.
(b) invariant under a tranlastion by a constant vector a^α.

Here, $u^\alpha \equiv P^\alpha / (-P^\beta P_\beta)^{1/2}$ is a 4-velocity of the center of mass and $P^\alpha \equiv \int T^{\alpha 0} \mathrm{d}^3 x$ is total momentum.

▶ **Solution:** (a) This is obvious since S_α only depends on invariant charges $J^{\alpha\beta}$ and P^α.

(b) While u^α does not change under constant translation, $J^{\alpha\beta}$ does. However, the change of spin 4-vector

$$
\delta S^\alpha = -\frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} \left(a^\beta P^\gamma - a^\gamma P^\beta \right) u^\delta = -\sqrt{-P^\eta P_\eta}\, \epsilon_{\alpha\beta\gamma\delta} a^\beta u^\gamma u^\delta = 0, \tag{5.6}
$$

vanishes due to antisymmetry of $\epsilon_{\alpha\beta\gamma\delta}$.

E3 Show that a spin 4-vector is orhtogonal to its 4-velocity u^α.

▶ **Solution:**

$$S_\alpha u^\alpha = -\frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}J^{\beta\gamma}u^\delta u^\alpha = 0 \tag{5.7}$$

since $\epsilon_{\alpha\beta\gamma\delta} = -\epsilon_{\delta\beta\gamma\alpha}$.

E4 Show that for the angular momentum in the center of mass frame it holds $J^{\alpha\beta}u_\beta = 0$:

▶ **Solution:** In center of mass frame we have

$$0 = \int x^i T^{00} \mathrm{d}^3 x, \qquad P^i = \int T^{0i} \mathrm{d}^3 x = 0. \tag{5.8}$$

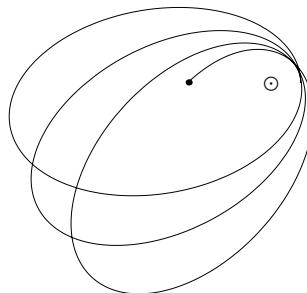(Notice that the second equality is a time derivative of the first). Therefore, in the center of mass frame $u_i = 0, u_0 = -1$ and

$$J^{0\beta}u_\beta = -J^{00} = 0 \tag{5.9}$$

$$J^{i\beta}u_\beta = -J^{i0} = -\int \left(x^i T^{00} - t T^{i0}\right)\mathrm{d}^3 x = 0. \tag{5.10}$$

E5 Show that a spin vector of a gyroscope, which is free of any external forces obey the equation for Fermi-Walker transport.

6

Geodesic motion

Test particles follow the geodesic lines of given spacetime. In order to resolve their motion one has to solve the geodesic equations. The constants of motion are connected with corresponding *Killing* field and is equivalent to statment that if metric does not depend on particular coordinate explicitly then there is an integral of motion, associated with that coordinate. Those concepts are practised in this section, devoted to the geodesic motion [1, 2, 3, 9].

E1 Show that the Hamiltonian for free test particle taken as

$$H = \frac{1}{2} g^{ij} p_i p_j \tag{6.1}$$

leads to the geodesic equation.

▶ **Solution:** The geodesic equation is derived from the Hamilton's equations

$$\frac{\mathrm{d} p_k}{\mathrm{d}\tau} = -\frac{\partial H}{\partial x^k}, \quad \frac{\mathrm{d} x^k}{\mathrm{d}\tau} = \frac{\partial H}{\partial p_k}. \tag{6.2}$$

Substituting for the Hamiltonian H we get

$$\frac{\mathrm{d} p_k}{\mathrm{d}\tau} = -\frac{1}{2} \frac{\partial g^{ij}}{\partial x^k} p_i p_j. \tag{6.3}$$

At the same time, however, it holds

$$\frac{\mathrm{d} p_k}{\mathrm{d}\tau} = \frac{\mathrm{d}}{\mathrm{d}\tau} \left(g_{ks} p^s \right) = \frac{\mathrm{d} p^s}{\mathrm{d}\tau} g_{ks} + p^s \frac{\partial g_{ks}}{\partial x^i} p^i. \tag{6.4}$$

Here we used the second Hamilton's equation, which in our case leads to

$$\frac{\mathrm{d}x^k}{\mathrm{d}\tau} = \frac{1}{2}g^{ij}\frac{\partial p_i}{\partial p_k}p_j + \frac{1}{2}g^{ij}p_i\frac{\partial p_j}{\partial p_k} = g^{kj}p_j = p^k. \tag{6.5}$$

Combining equations (6.3) and (6.4) we obtain

$$\begin{aligned}
g_{ks}\frac{\mathrm{d}p^s}{\mathrm{d}\tau} &= -\frac{1}{2}\frac{\partial g^{ij}}{\partial x^k}p_i p_j - \frac{\partial g_{ks}}{\partial x^i}p^i p^s \\
&= \frac{1}{2}g^{il}g^{sj}\frac{\partial g_{ls}}{\partial x^k}p_i p_j - \frac{\partial g_{ks}}{\partial x^i}p^i p^s. \tag{6.6}
\end{aligned}$$

At this point it is necessary to recall the following identity:

$$\frac{\partial}{\partial x^k}\left(g^{ij}g_{js}\right) = \frac{\partial \delta^i_s}{\partial x^k} = 0 \Rightarrow \frac{\partial g^{il}}{\partial x^k} = -g^{ij}g^{sl}\frac{\partial g_{js}}{\partial x^k}. \tag{6.7}$$

The equation (6.6) leads us to

$$\begin{aligned}
g_{ks}\frac{\mathrm{d}p^s}{\mathrm{d}\tau} &= \frac{1}{2}g_{ls,k}p^l p^s - \frac{1}{2}g_{ks,i}p^i p^s - \frac{1}{2}g_{ks,i}p^i p^s \\
&= |i \longleftrightarrow l| = \frac{1}{2}g_{ls,k}p^l p^s - \frac{1}{2}g_{ks,l}p^l p^s - \frac{1}{2}g_{ks,l}p^l p^s \\
&= |\text{in 3rd term } l \longleftrightarrow s| = \frac{1}{2}g_{ls,k}p^l p^s - \frac{1}{2}g_{ks,l}p^l p^s - \frac{1}{2}g_{kl,s}p^s p^l \\
&= \frac{1}{2}\left(g_{ls,k} - g_{ks,l} - g_{kl,s}\right)p^l p^s. \tag{6.8}
\end{aligned}$$

At the end, we clearly obtain

$$\frac{\mathrm{d}p^i}{\mathrm{d}\tau} + \frac{1}{2}g^{ik}\left(-g_{ls,k} + g_{ks,l} + g_{kl,s}\right)p^l p^s = 0 \rightarrow \frac{\mathrm{d}p^i}{\mathrm{d}\tau} + \Gamma^i_{ls}p^l p^s = 0. \tag{6.9}$$

E2 Determine geodesic equations for a test particle on 2-sphere with the line element

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2. \tag{6.10}$$

▶ **Solution:** We will use the result of the previous exercise and find the equations of motion from Hamilton's equations. Hamiltonian of a free test particle on a 2-sphere is given as

$$H = \frac{1}{2}g^{ij}p_i p_j = \frac{1}{2}(p_\theta)^2 + \frac{1}{2\sin^2\theta}(p_\varphi)^2. \tag{6.11}$$

The Hamilton's equations read

$$\frac{\mathrm{d}p_\theta}{\mathrm{d}\tau} = -\frac{\partial H}{\partial \theta} = \frac{\cot \theta}{\sin^2 \theta}(p_\varphi)^2 = \cot \theta \sin^2 \theta (p^\varphi)^2. \tag{6.12}$$

At the same time, it holds $p_\theta = p^\theta$ and, hence, we obtain the θ-component of the geodesic equation in the form

$$\boxed{\frac{\mathrm{d}p^\theta}{\mathrm{d}\tau} = \cot \theta \sin^2 \theta (p^\varphi)^2.} \tag{6.13}$$

For the φ-component of momentum we have

$$\frac{\mathrm{d}p_\varphi}{\mathrm{d}\tau} = -\frac{\partial H}{\partial \varphi} = 0 \Rightarrow p_\varphi = \mathrm{const} \equiv K = \sin^2 \theta p^\varphi. \tag{6.14}$$

The resulting equation is thus

$$\boxed{p^\varphi = \frac{1}{\sin^2 \theta}K.} \tag{6.15}$$

E3 Vacuum, static and spherically symmetric solution of the Einstein's equations is given by Schwarzschild metric which in Schwarzschild coordinates reads

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{c^2 r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2 \sin^2 \theta \mathrm{d}\varphi^2. \tag{6.16}$$

Find integrals of motion and in Newtonian limit determine their physical meaning.

▶ **Solution:** The Hamiltonian of a free test particle in the spacetime with metric $g_{\mu\nu}$ is given as

$$H = \frac{1}{2}g^{\mu\nu}p_\mu p_\nu. \tag{6.17}$$

It follows from Hamilton's equations

$$\frac{\mathrm{d}p_\alpha}{\mathrm{d}\tau} = -\frac{\partial H}{\partial x^\alpha} \tag{6.18}$$

for p_t and p_φ that

$$\frac{\mathrm{d}p_t}{\mathrm{d}\tau} = 0 \text{ and } \frac{\mathrm{d}p_\varphi}{\mathrm{d}\tau} = 0, \tag{6.19}$$

since the Hamiltonian H does not explicitly depends on t and φ. Thus, we have found two integrals of motion, namely

$$p_t = K_1 \text{ a } p_\varphi = K_2. \tag{6.20}$$

In the limit $r \to \infty$, Schwarzschild spacetime becomes Minkowski spacetime (we say, that Schwarzschild spacetime is asymptotically flat). The components of 4-momentum has the following interpretation:

$$p^\alpha = m\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} = m\gamma\frac{\mathrm{d}x^\alpha}{\mathrm{d}t} \tag{6.21}$$

where $\gamma = 1/\sqrt{1 - V^2/c^2}$ a $V^2 = [V^r]^2 + [V^\theta]^2 + [V^\varphi]^2$. From this we obtain

$$p^t = m\gamma c\frac{\mathrm{d}t}{\mathrm{d}t} = \gamma mc = E/c, \tag{6.22}$$

and hence

$$K_1 = p_t = -p^t = -E/c. \tag{6.23}$$

For the second integral of motion it clearly holds:

$$p^\varphi = \gamma m\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \gamma m\frac{r^2}{r^2}\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \gamma\frac{L}{r^2}. \tag{6.24}$$

If $V \ll c$ we have $\gamma \approx 1$ and so

$$K_2 = p_\varphi = r^2 p^\varphi = L. \tag{6.25}$$

$\boxed{\text{E4}}$ Show that motion in central plane of Schwarzschild spacetime is stable agains latitudal perturbations.

▶ **Solution:** The equation of motion for the latitudal coordinate is given as

$$\frac{\mathrm{d}p^\theta}{\mathrm{d}\tau} = \frac{\cos\theta}{r^4\sin^3\theta}L^2 - \frac{2}{r}p^r p^\theta. \tag{6.26}$$

Without the loss of generality we can assume that the motion takes place in the plane $\theta = \pi/2$ a $r = r_0 = \text{const}$. Let us further assume a small perturbation $\delta\theta$. Let us seek the solution of the equation (6.26) in the form $\theta = \pi/2 + \delta\theta$. We obtain, to the first order in $\delta\theta$

$$\ddot{\delta\theta} = -\frac{L^2}{r_0^4}\delta\theta, \tag{6.27}$$

where we used the identities

$$\cos(\pi/2 + \delta\theta) = \cos(\pi/2) - \sin(\pi/2)\delta\theta + \ldots \approx -\delta\theta. \qquad (6.28)$$

and

$$\sin(\pi/2 + \delta\theta) = \sin(\pi/2) + \cos(\pi/2)\delta\theta - \frac{1}{2}\sin(\pi/2)\delta\theta^2 + \ldots \approx 1. \quad (6.29)$$

The solution of the equation (6.27) is harmonic oscillations, i.e.

$$\delta\theta \sim \sin\tau. \qquad (6.30)$$

It follows that the of motion in the central plane is stable against latitudal perturbations.

E5 Using Hamilton-Jacobi equations determine the equations of motion of a test particle in Schwarzschild spacetime.

▶ **Solution:** Hamilton-Jacobi (HJ) equations for the S field reads

$$-\frac{\partial S}{\partial \lambda} = \frac{1}{2}g^{\alpha\beta}\frac{\partial S}{\partial x^\alpha}\frac{\partial S}{\partial x^\beta}, \qquad (6.31)$$

where

$$p_\alpha \equiv \frac{\partial S}{\partial x^\alpha}. \qquad (6.32)$$

Taking into account the normalization condition of 4-momentum $g^{\alpha\beta}p_\alpha p_\beta = -m^2$ we obtain a relation

$$\frac{\partial S}{\partial \lambda} = \frac{1}{2}m^2. \qquad (6.33)$$

In Schwarzschild spacetime the HJ equations are given as

$$-\frac{\partial S}{\partial \lambda} = -\frac{1}{2}f(r)^{-1}\left(\frac{\partial S}{\partial t}\right)^2 + \frac{f(r)}{2}\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{2r^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{1}{2r^2\sin^2\theta}\left(\frac{\partial S}{\partial \varphi}\right)^2.$$
$$(6.34)$$

Let us look for the solution in separable form

$$S = S_\lambda + S_t + S_r + S_\theta + S_\varphi. \qquad (6.35)$$

In previous exercises we determined two constants of motion, namely $p_t = -E$ and $p_\varphi = L$. Hence, from equations (6.32) and (6.33) we get

$$p_t = \frac{\partial S}{\partial t} = \frac{\partial S_t}{\partial t} \to S_t = -Et, \qquad (6.36)$$

$$p_\varphi = \frac{\partial S}{\partial \varphi} = \frac{\partial S_\varphi}{\partial \varphi} \to S_\varphi = L\varphi, \qquad (6.37)$$

$$\frac{1}{2}m^2 = \frac{\partial S}{\partial \lambda} = \frac{\partial S_\lambda}{\partial \lambda} \to S_\lambda = \frac{1}{2}m^2\lambda. \qquad (6.38)$$

Using these, the solution (6.35) can be rewritten as

$$S = \frac{1}{2}m^2\lambda - Et + L\varphi + S_r + S_\theta. \tag{6.39}$$

We plug this solution back into (6.34) to obtain

$$-\frac{1}{2}m^2 = -\frac{1}{2}f(r)^{-1}E^2 + \frac{1}{2}f(r)\left(\frac{\partial S_r}{\partial r}\right)^2 + \frac{1}{2r^2}\left(\frac{\partial S_\theta}{\partial \theta}\right)^2 + \frac{1}{2r^2\sin^2\theta}L^2. \tag{6.40}$$

After a simple algebra and separation of r-dependent and θ-dependent parts we obtain

$$\frac{r^2}{f(r)}E^2 - f(r)r^2\left(\frac{\partial S_r}{\partial r}\right)^2 - m^2r^2 = \left(\frac{\partial S_\theta}{\partial \theta}\right) + \frac{L^2}{\sin^2\theta} = K = \text{const.} \tag{6.41}$$

Substituting the expressions for $p_r = \partial S_r/\partial r$ and $p_\theta = \partial S_\theta/\partial \theta$ we obtain two equations:

$$(p^r)^2 = E^2 - f(r)\left(m^2 + \frac{K}{r^2}\right), \tag{6.42}$$

$$\left(p^\theta\right)^2 = \frac{1}{r^4}\left(K - \frac{L^2}{\sin^2\theta}\right). \tag{6.43}$$

If we consider the motion in equatorial plane $\theta = \pi/2$, $p^\theta = 0$ we have for a latitudal component of 4-momentum

$$0 = K - L^2. \tag{6.44}$$

This motivates to introduce a new constant of motion, namely $Q = K - L^2$ for which it holds $Q = 0$ in equatorial plane. Rewriting (6.42) and (6.43) with this constant of motion in mind we obtain

$$(p^r)^2 = E^2 - f(r)\left(m^2 + \frac{L^2 + Q}{r^2}\right), \tag{6.45}$$

$$\left(p^\theta\right)^2 = \frac{1}{r^4}\left(Q + L^2 - \frac{L^2}{\sin^2\theta}\right). \tag{6.46}$$

The remaining equations for time and azimuthal coordinates are clearly

$$p^t = -\frac{p_t}{f(r)} = \frac{E}{f(r)}, \tag{6.47}$$

$$p^\varphi = \frac{1}{r^2\sin^2\theta}p_\varphi = \frac{L}{r^2\sin^2\theta}. \tag{6.48}$$

Problem set for relativistic physics and astrophysics

E6 Using the Lagrange formalism discuss the motion of a test particle and photons in equatorial plane of (outer) Schwarzschild spacetime, determine the conditions for the existence of circular orbits and investigate their stability via the effective potential $V_{\text{eff}}(r; L)$.

▶ **Solution:** Let us start from the equation of motion for the radial component of 4-velocity of a test particle.

$$[u^r]^2 = E^2 - V_{\text{eff}}(r; L). \tag{6.49}$$

Differentiating it with respect to affine parameter τ we obtain

$$\frac{\mathrm{d}u^r}{\mathrm{d}\tau} = -\frac{1}{2}\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}r}. \tag{6.50}$$

On circular orbit it clearly holds $u^r = 0$ and $\mathrm{d}u^r/\mathrm{d}\tau = 0$. Putting the second condition into (6.50) we obtain a formula for determining circular orbit from the effective potential as

$$\boxed{\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}r} = 0.} \tag{6.51}$$

The stability of circular orbit r_0 can be investigated via considering a small perturbation δr. In other words we take $r = r_0 + \delta r$. Plugging this into equation (6.50) we get

$$
\begin{aligned}
\ddot{\delta r} &= -\frac{1}{2}\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}r} = |\text{Taylor series}| \\
&= -\frac{1}{2}\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}r}\big|_{r0} - \frac{1}{2}\frac{\mathrm{d}^2 V_{\text{eff}}}{\mathrm{d}r^2}\big|_{r0}\delta r + ... \\
&\approx -\frac{1}{2}\frac{\mathrm{d}^2 V_{\text{eff}}}{\mathrm{d}r^2}\big|_{r0}\delta r.
\end{aligned}
\tag{6.52}
$$

If

$$\frac{\mathrm{d}^2 V_{\text{eff}}}{\mathrm{d}r^2}\big|_{r0} > 0 \text{ (local minimum)}, \tag{6.53}$$

we see that δr undergoes harmonic oscillation. Thus, the perturbation δr is bounded, which implies **stability** of the corresponding circular orbit.

On the other hand, if

$$\frac{\mathrm{d}^2 V_{\text{eff}}}{\mathrm{d}r^2}\big|_{r0} < 0 \text{ (local maximum)}, \tag{6.54}$$

we obtain an exponential growth of the perturbation δr, which implies **instability** of the circular orbit.

E7 Determine the angle of deflection of a light ray in Schwarzschild space-time. Assume that the motion of the ray is confined to the equatorial plane ($\theta = \pi/2$) and that the minimal distance from the center is $r_{\min} \gg 2$.

▶ **Solution:** From the normalization condition of null equatorial geodesics in Schwatzschild spacetime:

$$0 = -\left(1 - \frac{2}{r}\right)[u^t]^2 + \left(1 - \frac{2}{r}\right)^{-1}[u^r]^2 + r^2[u^\varphi]^2, \tag{6.55}$$

we learn that the equation for radial component of a 4-velocity has a form

$$[u^r]^2 = \left[1 - \left(1 - \frac{2}{r}\right)\frac{b^2}{r^2}\right]u_t^2, \tag{6.56}$$

where $b = -u_\varphi/u_t = L/E$ is the so-called impact parameter. The equation for azimuthal component of 4-velocity is simply

$$u^\varphi = \frac{b}{r^2}u_t. \tag{6.57}$$

Introducing the reciprocal coordinate $u = 1/r$ and combining equations (6.56) and (6.57) we otain the following:

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2 = \frac{1}{b^2} - (1 - 2u)u^2. \tag{6.58}$$

Differentiating this equation with respect to φ we get

$$u'' = -u + 3u^2, \tag{6.59}$$

where we denoted $' \equiv \mathrm{d}/\mathrm{d}\varphi$.

For null geodesics in Minkowski spacetime it holds that

$$u_0'' = -u_0, \tag{6.60}$$

solution of which we take in the form

$$u_0 = A\sin\varphi + B\cos\varphi. \tag{6.61}$$

From the initial conditions we determine constants A and B. The first detivative of u_0 is:

$$(u_0')^2 = (A\cos\varphi - B\sin\varphi)^2 = \frac{1}{b^2} - u_0^2. \tag{6.62}$$

For $u_0 = 0$ (foton at infinity) and $\varphi_0 = 0$ we, therefore, get $A = 1/b$ and from equation (6.61) we obtain $B = 0$. Thus, the solution in asymptotically flat infinity $(r \to \infty)$ is given as

$$u_0 = \frac{1}{b} \sin \varphi. \tag{6.63}$$

Let us consider the solution of the equation (6.59) for finite distance in the form

$$u = u_0 + u_1, \tag{6.64}$$

where $u_1 \ll 1$. To the first order in u_1 we obtain the equation

$$u_1'' = -u_1 + \frac{3}{b^2} \sin^2 \varphi. \tag{6.65}$$

Instead of solving this, we transform the above into an equivalent form

$$u_1'' + u_1 = +\frac{3}{2b^2}(1 - \cos 2\varphi). \tag{6.66}$$

Its particular solution can be ascertained in the form

$$u_1 = \frac{3}{2b^2} - \frac{3k}{2b^2} \cos 2\varphi. \tag{6.67}$$

Plugging this into (6.66) give us the following condition

$$k = -\frac{1}{3}. \tag{6.68}$$

The result is thus

$$u = \frac{1}{b} \sin \varphi + \frac{3}{2b^2} + \frac{1}{2b^2} \cos 2\varphi. \tag{6.69}$$

The angle of deflection is defined via:

$$\delta = |\varphi_1(u \to 0) - \varphi_2(u \to 0)| - \pi. \tag{6.70}$$

Let

$$\varphi_1 = \pi + \epsilon_1 \tag{6.71}$$

and

$$\varphi_2 = \epsilon_2. \tag{6.72}$$

Substituting into (6.69) we obtain the equation

$$0 \simeq -\frac{1}{b}\epsilon_1 + \frac{3}{2b^2} + \frac{1}{2b^2}, \tag{6.73}$$

$$0 \simeq \frac{1}{b}\epsilon_2 + \frac{3}{2b^2} + \frac{1}{2b^2}, \tag{6.74}$$

from which we learn for ϵ_1 and ϵ_2 the following:

$$\epsilon_1 \simeq \frac{2}{b}, \quad \epsilon_2 \simeq -\frac{2}{b}. \tag{6.75}$$

The angle of deflection is thus

$$\boxed{\delta = |\epsilon_1 - \epsilon_2| \simeq \frac{4}{b}.} \tag{6.76}$$

E8 Determine the relationship for the magnitude of precesion of the pericenter for a test particle moving in the Schwarzschild spacetime.

▶ **Solution:** As it was show previously, in a spherically symmetric and static spacetime the motion of a test particle takes place in central plane. Without the loss of generality we can orient our coordinate system in such a way that the corresponding central plane is the equatorial plane, i.e. $\theta = \pi/2$.

First, let us derive the relativistic version of the Binet's formula. In the equatorial plane the relationship for the radial component of the 4-velocity U^r is given as

$$[U^r]^2 = E^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{L^2}{r^2}\right), \tag{6.77}$$

where M is a mass parameter of the Schwarzschild's spacetime, $E = -u_t$ is a covariant specific energy and $L = u_\varphi$ is the specific angular moment of of the test particle. Substituting $r = 1/u$ we change (6.77) into the

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\tau}\right)^2 = u^4\left[E^2 - (1 - 2Mu)(1 + L^2u^2)\right], \tag{6.78}$$

where τ is the proper time measured along the geodesic of the test particle. Together with the relation

$$\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right)^2 = \frac{L^2}{r^4} = L^2u^4 \tag{6.79}$$

we obtain differential equation for $u = u(\varphi)$ in the form

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2 = \frac{1}{L^2}\left[E^2 - (1 - 2Mu)(1 + L^2u^2)\right]. \tag{6.80}$$

Differentiating this equation and after subsequent algebra we arrive at the *Binet's formula*:

$$\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} + u = \frac{M}{L^2} + 3Mu^2. \tag{6.81}$$

The first term on the right-hand side corresponds to the Newtonian term, while the second term represents relativistic correction. In case of Mercury, for example, the ration between the first and second term is

$$\frac{3Mu^2}{M/L^2} = 3L^2u^2 = \frac{3L^2}{r^2} \sim 10^{-7}. \tag{6.82}$$

To solve (6.81) we use perturbation method. Let us first introduce a parameter

$$\epsilon = \frac{3M^2}{L^2} \tag{6.83}$$

and we rewrite (6.81) into the form

$$\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} + u = \frac{M}{L^2} + \epsilon\left(\frac{L^2}{M}u^2\right). \tag{6.84}$$

Further, let us consider an ansatz

$$u = u_0 + \epsilon u_1, \tag{6.85}$$

where $|u_1| \ll 1$ and u_0 is the solution to the classical (non-relativistic) Binet's formula, which is a cone-section:

$$u_0 = \frac{M}{L^2}(1 + e\cos\varphi). \tag{6.86}$$

Substituting (6.85) into (6.84) we obtain the equation

$$\begin{aligned}
u_0'' + \epsilon u_1'' + u_0 + \epsilon u_1 &= \frac{M}{L^2} + \epsilon\left(\frac{L^2}{M}\right)(u_0^2 + 2u_0u_1\epsilon + \epsilon^2u_1^2) \\
&\simeq \frac{M}{L^2} + \epsilon\left(\frac{L^2}{M}u_0^2\right). \tag{6.87}
\end{aligned}$$

Since

$$u_0'' + u_0 = \frac{M}{L^2}, \tag{6.88}$$

we are left with

$$u_1'' + u_1 = \frac{M}{L^2}\left(1 + \frac{e^2}{2}\right) + \frac{2M}{L^2}e\cos\varphi + \frac{Me^2}{2L^2}\cos 2\varphi. \tag{6.89}$$

A particular solution of (6.89) can be find in the form

$$u_1 = A + B\varphi\sin\varphi + C\cos 2\varphi. \tag{6.90}$$

After substituting this into (6.89) we have

$$A + 2B\cos\varphi - 3C\cos 2\varphi = \frac{M}{L^2}\left(1 + \frac{e^2}{2}\right) + \frac{2M}{L^2}e\cos\varphi + \frac{Me^2}{2L^2}\cos 2\varphi. \tag{6.91}$$

From this, we learn the following relations between parameters A, B and C:

$$A = \frac{M}{L^2}\left(1 + \frac{e^2}{2}\right), \tag{6.92}$$

$$B = \frac{Me}{L^2}, \tag{6.93}$$

$$C = -\frac{Me^2}{6L^2}. \tag{6.94}$$

Then, the solution u reads

$$u = u_0 + \epsilon\frac{M}{L^2}\left(1 + \frac{e^2}{2} + e\varphi\sin\varphi - \frac{e^2}{6}\cos 2\varphi\right). \tag{6.95}$$

The largest correction to u_0 comes from $\varphi\sin\varphi$, since it grows with increasing φ. If we neglect the remaining terms the solution can be written as

$$u \simeq \frac{M}{L^2}\left[1 + e(\cos\varphi + \epsilon\varphi\sin\varphi)\right]. \tag{6.96}$$

Since it holds that

$$\cos(\varphi - \epsilon\varphi) = \cos\varphi\underbrace{\cos(\epsilon\varphi)}_{\sim 1} + \sin\varphi\underbrace{\sin(\epsilon\varphi)}_{\sim\epsilon\varphi} \simeq \cos\varphi + \epsilon\varphi\sin\varphi, \tag{6.97}$$

we arrive at the final form

$$u \simeq \frac{M}{L^2} \left(1 + e \, \cos[\varphi(1 - \epsilon)]\right). \tag{6.98}$$

The phase difference corresponding to one period is given as

$$\varphi(1 - \epsilon) = 2\pi \Rightarrow \varphi_p = \frac{2\pi}{1 - \epsilon} \simeq 2\pi(1 + \epsilon). \tag{6.99}$$

Thus, the angle determining the magnitude of the precession is given by

$$\boxed{\Delta\varphi = \varphi_p - 2\pi \simeq 2\pi\epsilon = \frac{6\pi M^2}{L^2}.} \tag{6.100}$$

E9* The notion of circular orbits is very useful in understanding the dynamics of particles in curved spacetimes. There are two key circular orbits marginally stable r_{ms}, and photon r_{ph} orbits. Let $V(r)$ is the effective potential of test particle in R-N spacetime and write it in the form

$$V(r) = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)\left(b + \frac{L_z^2}{r^2}\right) \tag{6.101}$$

where $b = 1$ for massive test particles and $b = 0$ for mass-less particles. Using following conditions find out radii of those orbits as function of electric charge parameter Q.

(A) The Marginally stable orbit is determined by the condition

$$\frac{\mathrm{d}V}{\mathrm{d}r} = 0 \quad \text{and} \quad \frac{\mathrm{d}^2V}{\mathrm{d}r^2} = 0. \tag{6.102}$$

(B) The Photon circular orbit satisfies condition

$$\frac{\mathrm{d}V}{\mathrm{d}r} = 0 \quad \text{and} \quad \frac{L^2}{E^2} = \frac{r^2}{1 - 2M/r + Q^2/r^2}. \tag{6.103}$$

7

Black holes

The region where even the light cannot escape from is called the event horizon and is the boundary that defines the extension of the object we call the black hole. The passage of time and tidal forces as experienced by an observer falling down towards physical singularity are examined in this section [1, 9].

$\boxed{\text{E1}}$ Let us consider a test particle which radially falls from $r = R$ on Schwarzschild black hole. Determine the interval of the proper time $\Delta\tau$ which it takes the particle to arrive at singularity $(r = 0)$ and the interval of the coordinate time Δt to arrive at the horizon $(r = 2)$. The spacetime line element has in the Schwarzschild's coordinates the form

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{1}{f(r)}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2, \qquad (7.1)$$

where $f(r) = 1 - 2/r$.

▶ **Solution:** Radially falling particle has the following components of 4-velocity $U^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$:

$$U = \left(U^t, U^r, 0, 0\right), \qquad (7.2)$$

where the 4-velocity is nomalized as

$$-1 = g_{\mu\nu}U^\mu U^\nu \Rightarrow (U^r)^2 = E^2 - (1 - 2/r). \qquad (7.3)$$

Here, we have used the existence of the integral of motion $U_t = -E$ which is interpreted as covariant specific energy measured by static observers at

infinity. The particle at $r = R$ is released at rest which means that the corresponding value of E^2 is

$$E^2 = 1 - 2/R. \tag{7.4}$$

For the interval of proper time of a falling particle we hence obtain the equation

$$\Delta\tau = -\int_R^0 \frac{\mathrm{d}r}{\sqrt{2/r - 2/R}}, \tag{7.5}$$

where the sign '−' means that the particle falls towards the black hole. The integral (7.5) is solved as follows:

$$
\begin{aligned}
\Delta\tau &= \sqrt{\frac{R}{2}} \int_0^R \sqrt{\frac{r/R}{1 - r/R}} = |r/R = \cos^2\eta, \quad \mathrm{d}r = -2R\cos\eta\sin\eta\,\mathrm{d}\eta| \\
&= \frac{2}{\sqrt{2}} R^{3/2} \int_0^{\pi/2} \cos^2\eta\,\mathrm{d}\eta = |\cos^2\eta = (1 + \cos 2\eta)/2| \\
&= \frac{2}{\sqrt{2}} R^{3/2} \int_0^{\pi/2} \left(\frac{1}{2} + \frac{1}{2}\cos 2\eta\right)\mathrm{d}\eta = \left(\frac{R}{2}\right)^{3/2}\pi.
\end{aligned} \tag{7.6}
$$

To determine the coordinate time Δt corresponding to the proper time of a static observer at infinity we use the equation (7.3) and the relation $U^t = g^{tt}U_t$. We obtain

$$\frac{U^r}{U^t} = \frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{(1 - 2/r)\sqrt{E^2 - (1 - 2/r)}}{E}. \tag{7.7}$$

This differential equation can be solved simply by integration:

$$
\begin{aligned}
\Delta t &= -\int_R^2 \frac{E\,\mathrm{d}r}{(1 - 2/r)\sqrt{E^2 - (1 - 2/r)}} \\
&= -\int_R^2 \frac{\sqrt{1 - 2/R}\,\mathrm{d}r}{(1 - 2/r)\sqrt{2/r - 2/R}}.
\end{aligned} \tag{7.8}
$$

In analogy to the previous case we use the substitution

$$r/R = \cos^2\eta \tag{7.9}$$

with which (7.8) turns into

$$
\Delta t = \sqrt{\frac{2(1-2/R)}{R}} \int_0^{\eta_0} \frac{\cos^4 \eta}{\sin^2 \eta - 2/R} \mathrm{d}\eta
$$

$$
= \sqrt{\frac{2(1-2/R)}{R}} \times
$$

$$
\left[\frac{(4+R)\eta}{2R} + \frac{2\sqrt{2}\,\mathrm{arctanh}\left(\frac{\sqrt{2}\,\mathrm{tg}\,\eta}{\sqrt{R-2}}\right)}{R\sqrt{R-2}} + \frac{1}{2}\cos\eta\sin\eta \right]_0^{\arccos\sqrt{2/R}} \quad (7.10)
$$

The dominant term is $\mathrm{arctanh}(x)$ as its argument for the upper limit $\eta = \arccos\sqrt{2/R}$ corresponds to $x = 1$ for which the function arctanh diverges. Thus means that from the point of view of a coordinate observer (a static observer at infinity) the test particle never reaches the horizon of the black hole, while from the point of view of the particle itself the singularity (and thus the horizon as well) is reached in a finite time.

E2 Determine the components of tidal acceleration experienced by radially in-falling observer in Schwarzschild black hole spacetime.

▶ **Solution:** We start with the equation of geodesic deviation which has the form

$$
\frac{\mathrm{D}^2 l^a}{\mathrm{d}\tau^2} = -R^a{}_{bcd} U^b l^c U^d, \quad (7.11)
$$

where $\mathbf{R}$ is a Riemann tensor, $\mathbf{U}$ is the 4-velocity of the in-falling observer and $\mathbf{l}$ is the so-called separation vector. We want to find components of tital acceleration $\delta\mathbf{a} = \mathrm{D}^2\mathbf{l}/\mathrm{d}\tau^2$ felt by the unlucky observer which is falling towards the black hole. Hence, we transform the equation (7.11) into an orthonormal system attached to the freely falling observer, i.e.

$$
\frac{\mathrm{D}^2 l^{\hat{a}}}{\mathrm{d}\tau^2} = -R^{\hat{a}}{}_{\hat{b}\hat{c}\hat{d}} U^{\hat{b}} l^{\hat{c}} U^{\hat{d}}. \quad (7.12)
$$

In this system, the separation vector has the zero component $l^{\hat{0}} = 0$ and the remaining components are $l^{\hat{1}} = \delta r$, $l^{\hat{2}} = \delta\theta$ a $l^{\hat{3}} = \delta\varphi$. The zero components of the 4-velocity is obviously $U^{\hat{0}} = 1$ and the remaining components are zero. The transformation from (7.11) to (7.12) can be made in two steps: first, we switch from coordinate base to the orthonormal

base of a static observer

$$\mathbf{e}_{\hat{t}} = (1 - 2/r)^{-1/2}\mathbf{e}_t, \tag{7.13}$$

$$\mathbf{e}_{\hat{r}} = (1 - 2/r)^{1/2}\mathbf{e}_r, \tag{7.14}$$

$$\mathbf{e}_{\hat{\theta}} = r^{-1}\mathbf{e}_\theta, \tag{7.15}$$

$$\mathbf{e}_{\hat{\varphi}} = (r\sin\theta)^{-1}\mathbf{e}_\varphi \tag{7.16}$$

and subsequently by Lorentz transformation (a.k.a. 'boost') in the radial direction we change the basis to the orthonormal basis of a freely falling observer

$$\mathbf{e}_{\hat{0}} = \mathbf{U} = \gamma\mathbf{e}_{\hat{t}} - \gamma v\mathbf{e}_{\hat{r}}, \tag{7.17}$$

$$\mathbf{e}_{\hat{1}} = -\gamma v\mathbf{e}_{\hat{t}} + \gamma\mathbf{e}_{\hat{r}}, \tag{7.18}$$

$$\mathbf{e}_{\hat{2}} = \mathbf{e}_{\hat{\theta}}, \tag{7.19}$$

$$\mathbf{e}_{\hat{3}} = \mathbf{e}_{\hat{\varphi}}. \tag{7.20}$$

The components of a Riemann tensor in static orthonormal frame can be ascertained, for instance, using the Cartan formalism (see chapter 3, Ex2, where we put $e^{2\alpha(r)} = e^{-2\beta(r)} = 1 - 2/r$). Hence:

$$R^{\hat{t}}_{\hat{r}\hat{t}\hat{r}} = \frac{2}{r^3}, \qquad R^{\hat{t}}_{\hat{\theta}\hat{t}\hat{\theta}} = R^{\hat{t}}_{\hat{\varphi}\hat{t}\hat{\varphi}} = -\frac{1}{r^3},$$

$$R^{\hat{r}}_{\hat{\theta}\hat{r}\hat{\theta}} = R^{\hat{r}}_{\hat{\varphi}\hat{r}\hat{\varphi}} = -\frac{1}{r^3}, \qquad R^{\hat{\theta}}_{\hat{\varphi}\hat{\theta}\hat{\varphi}} = \frac{2}{r^3}. \tag{7.21}$$

Contracting these components with the orthonormal tetrad (7.17)–(7.20) gives us their expression in the coordinate frame of a freely falling observer:

$$R^{\hat{a}}_{\hat{b}\hat{c}\hat{d}} = e^{\hat{a}}_{\hat{\alpha}}e^{\hat{\beta}}_{\hat{b}}e^{\hat{\gamma}}_{\hat{c}}e^{\hat{\delta}}_{\hat{d}}R^{\hat{\alpha}}_{\hat{\beta}\hat{\gamma}\hat{\delta}}, \tag{7.22}$$

where greek indices denote components in static orthonormal tetrad. A specific feature of Schwarzschild geometry is that they are invariant with respect to Lorentz transformation in radial direction, which can be easily checked. For instance, the component

$$R^{\hat{0}}_{\hat{1}\hat{0}\hat{1}} = \eta^{00}R_{\hat{0}\hat{1}\hat{0}\hat{1}} = -R_{\hat{0}\hat{1}\hat{0}\hat{1}} = -e^{\hat{\alpha}}_{\hat{0}}e^{\hat{\beta}}_{\hat{1}}e^{\hat{\gamma}}_{\hat{0}}e^{\hat{\delta}}_{\hat{1}}R_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}} =$$
$$- \left[\gamma^4 R_{\hat{t}\hat{r}\hat{t}\hat{r}} + (-\gamma v)^4 R_{\hat{r}\hat{t}\hat{r}\hat{t}} + \gamma^2(-\gamma v)^2 R_{\hat{t}\hat{r}\hat{r}\hat{t}} + \gamma^2(-\gamma v)^2 R_{\hat{r}\hat{t}\hat{t}\hat{r}} \right] =$$
$$-\gamma^4(1 - v^2)^2 R_{\hat{t}\hat{r}\hat{t}\hat{r}} = R^{\hat{t}}_{\hat{r}\hat{t}\hat{r}}. \tag{7.23}$$

Similarly

$$R^{\hat{0}}_{\hat{2}\hat{0}\hat{2}} = R^{\hat{t}}_{\hat{\theta}\hat{t}\hat{\theta}}, \quad R^{\hat{0}}_{\hat{3}\hat{0}\hat{3}} = R^{\hat{t}}_{\hat{\varphi}\hat{t}\hat{\varphi}}, \quad \dots \tag{7.24}$$

The resulting components of the tidal acceleration as measured by freely falling observer are

$$\frac{D^2 l^{\hat{j}}}{d\tau^2} = -R^{\hat{j}}_{\hat{0}\hat{k}\hat{0}} l^{\hat{k}}, \tag{7.25}$$

which gives us

$$\frac{D^2 l^{\hat{1}}}{D\tau^2} = \frac{2}{r^3} \delta r, \tag{7.26}$$

$$\frac{D^2 l^{\hat{2}}}{D\tau^2} = -\frac{1}{r^3} \delta\theta, \tag{7.27}$$

$$\frac{D^2 l^{\hat{3}}}{D\tau^2} = -\frac{1}{r^3} \delta\varphi. \tag{7.28}$$

Thus, we see that while in radial direction the acceleration is positive (resulting in stretching of the observer), in remaining two directions it is negative (resulting in contraction of the observer).

E3* Determine the magnitude and orientation of local electric and magnetic field in the vicinity of Reissner-Nordstrøm black hole of the mass M and charge Q as measured by the observer on a circular orbit with circumference $2\pi r$.

E4* Charged particles are radially in-falling into Reissner-Nordstrøm black hole with parameters $Q^2 < M^2$. Show that in this way it is impossible to turn the black hole into naked singularity, that is an object with parameters $Q^2 > M^2$.

**E5*** Consider an observer orbiting a Kerr black hole in equatorial plane on a circular orbit $r = \text{const}$.

(a) Determine the allowed range for angular velocity $\Omega = d\varphi/dt$ as measured stationary observers at infinity.

(b) If the orbit is located within the ergosphere, show that from the point of view of observers at infinity, the orbiting observer cannot be at rest and must orbit the black hole in the same direction as it spins.

(c) If the observer is between the horizons, show that the observer cannot stay at the orbit $r = \text{const}$.

Problem set for relativistic physics and astrophysics

E6* Consider the Sgr A^* black hole with mass $M_{Sgra^*} = 4.02 \times 10^6 M_\odot$ (although this black hole rotates, consider it as a non-rotating one). Determine:

(a) the magnitude curvature tensor at its horizon,

(b) the proper time of experimental capsule that falls from the horizon to Sgr A^*'s physical singularity,

(a) the moment of capsule's proper time τ when the tidal forces are of order of unity.

E7* Consider the star S_2 on circular orbit with radius $R_2 = 7.9\text{kpc}$ around Sgr A^* black hole with the velocity 7650km/s. The motion takes place in the equatorial plane of Sgr A^*. What would be the magnitude of spin parameter a of corresponding Kerr spacetime?

8
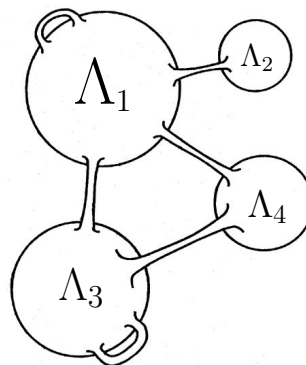
Cosmology

Einstein's field equations can be applied to the whole universe. The basic tool of a cosmologist are Friedmann equations, desribing dynamics of the Universe and Robertson-Walker metric responsible for description of the geometry of the Universe. These concepts and more are practised in this section [3, 6, 7, 9].

E1 Find a transformation which turns Robertson-Walker (R-W) metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + R^2(t)\left[\frac{\mathrm{d}r^2}{1 - kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2)\right] \tag{8.1}$$

into

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + R^2(t)\left[\mathrm{d}\chi^2 + \Sigma_k^2(\chi)(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2)\right], \tag{8.2}$$

where

$$\Sigma_k(\chi) = \begin{cases} \chi & \text{for} \quad k = 0, \\ \sin\chi & \text{for} \quad k = +1, \\ \sinh\chi & \text{for} \quad k = -1. \end{cases} \tag{8.3}$$

▶ **Solution:** The corresponding transformation is clearly

$$r = \Sigma_k(\chi). \tag{8.4}$$

For $k = 0$ we get $\mathrm{d}r = \mathrm{d}\chi$ and so

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + R^2(t)\left[\mathrm{d}\chi^2 + \chi^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2)\right]. \tag{8.5}$$

For $k = +1$ we have $\mathrm{d}r = \cos\chi\mathrm{d}\chi$, which together with the identity $\sin^2\chi + \cos^2\chi = 1$ leads to the result

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + R^2(t)\left[\mathrm{d}\chi^2 + \sin^2\chi^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2)\right]. \tag{8.6}$$

For $k = -1$ we have $\mathrm{d}r = \cosh\chi\mathrm{d}\chi$, which together with the identity $\cosh^2\chi - \sinh^2\chi = 1$ leads to

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + R^2(t)\left[\mathrm{d}\chi^2 + \sinh^2\chi^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2)\right]. \tag{8.7}$$

E2 Derive the formula for cosmological red shift

$$1 + z = \frac{R(t_0)}{R(t)}. \tag{8.8}$$

▶ **Solution:** Let us consider a galaxy $\mathcal{G}$ with radial co-moving coordinate χ along which an emitted photon is moving. From the condition for the worldline of an photon, i.e. $\mathrm{d}s^2 = 0$, we obtain

$$0 = -\mathrm{d}t^2 + R^2(t)\mathrm{d}\chi^2 \quad \Rightarrow \quad \chi = \int_{t_1}^{t_2} \frac{\mathrm{d}t}{R(t)}. \tag{8.9}$$

After passage of period Δt_1, another photon is emiited, which reach the observer in time $t_2 + \Delta t_2$. Thus, we have the equation

$$\chi = \int_{t_1}^{t_2} \frac{\mathrm{d}t}{R(t)} = \int_{t_1+\Delta t_1}^{t_2+\Delta t_2} \frac{\mathrm{d}t}{R(t)}, \tag{8.10}$$

which can be written as

$$\int_{t_1}^{t_1+\Delta t_1} \frac{\mathrm{d}t}{R(t)} + \int_{t_1+\Delta t_1}^{t_2} \frac{\mathrm{d}t}{R(t)} = \int_{t_1+\Delta t_1}^{t_2} \frac{\mathrm{d}t}{R(t)} + \int_{t_2}^{t_2+\Delta t_2} \frac{\mathrm{d}t}{R(t)}. \tag{8.11}$$

After subtracting same terms we get

$$\int_{t_2}^{t_2+\Delta t_2} \frac{\mathrm{d}t}{R(t)} = \int_{t_1}^{t_1+\Delta t_1} \frac{\mathrm{d}t}{R(t)}. \tag{8.12}$$

Since during the periods Δt_i the scale parameter is not changing noticeably we can say

$$R(t_1) \simeq R(t_1 + \Delta t_1) \quad \text{and} \quad R(t_2) \simeq R(t_2 + \Delta t_2). \tag{8.13}$$

After some algebra we obtain the result

$$\frac{R(t_2)}{R(t_1)} = \frac{\Delta t_2}{\Delta t_1} = \frac{f_1}{f_2}. \tag{8.14}$$

The red shift z is defined via relationship

$$1 + z = f_{\text{source}}/f_{\text{observer}}. \tag{8.15}$$

In our case the index 1 corresponds to the source, while index 2 corresponds to the observer. Thus

$$\frac{R(t_2)}{R(t_1)} = 1 + z. \tag{8.16}$$

E3 Determine the parameters of Einstein's static universe and show that that model is unstable.

▶ **Solution:** We start with Friedmann equations in the form

$$3\frac{\dot{R}^2 + k}{R^2} - \Lambda = 8\pi\varrho \tag{8.17}$$

$$\frac{2R\ddot{R} + \dot{R}^2 + k}{R^2} - \Lambda = -8\pi p. \tag{8.18}$$

Simplifying the first, we get

$$\dot{R}^2 = \frac{8\pi}{3}\varrho R^2 + \frac{\Lambda}{3}R^2 - k. \tag{8.19}$$

Substituting $\dot{R}^2$ into (8.18) we obtain

$$2R\ddot{R} = -8\pi p R^2 - \frac{8\pi}{3}\varrho R^2 + \frac{2}{3}\Lambda R^2. \tag{8.20}$$

In order for a universe to be static $\dot{R} = \ddot{R} = 0$ must holds. For simplicity, let us represent the matter with which the Universe is filled as dust, which is described by equation of state $p = 0$. Given this, from equations (8.19) and (8.20) we obtain

$$k = \Lambda R^2 \quad \text{a} \quad \varrho = \varrho_E = \frac{\Lambda}{4\pi}. \tag{8.21}$$

Since $\Lambda > 0$ (from condition $\varrho > 0$), the geometry of static universe has a positive curvature, i.e. $k = 1$. The radius of Einstein's universe is thus

$$R_E = \frac{1}{\sqrt{\Lambda}}. \tag{8.22}$$

Now it is necessary to adress the question of stability of our solution against small perturbation of density ϱ and of scale parameter R. For this purpose we define the following functions

$$R(t) = \frac{1}{\sqrt{\Lambda}} + \delta R(t) \quad \text{and} \quad \varrho(t) = \frac{\Lambda}{4\pi} + \delta\varrho(t), \tag{8.23}$$

which we put into (8.20) and retain the terms only up to first order in δR:

$$2\delta\ddot{R}\sqrt{\Lambda} \simeq -\frac{8\pi}{3}\delta\varrho. \tag{8.24}$$

The relationship between $\delta\varrho$ and δR for a dust follows from the condition of adiabatic expansion

$$\mathrm{d}(\varrho R^3) = 0 \quad \Rightarrow \quad \delta\varrho = -3\frac{\varrho_E}{R_E}\delta R = -\frac{3}{4\pi}\Lambda^{3/2}\delta R. \tag{8.25}$$

Thus, we get

$$\delta\ddot{R} = \Lambda\delta R. \tag{8.26}$$

Since $\Lambda > 0$, the solution of this equation is exponentially growing perturbation

$$\delta R \sim \exp(t) \quad \Rightarrow \quad \text{instability.} \tag{8.27}$$

E4 Derive the equation for conservation of energy during adiabatic expansion (compression) of the universe.

▶ **Solution:** The sought equation follows from divergencelessness of a energy-momentum tensor of an ideal fluid. In other words, we have

$$u_\mu \nabla_\nu T^{\mu\nu} = 0, \tag{8.28}$$

$$T^{\mu\nu} = (p + \varrho)u^\mu u^\nu + pg^{\mu\nu}. \tag{8.29}$$

Substituting (8.29) into (8.28) we obtain:

$$
\begin{aligned}
u_\mu \nabla_\nu T^{\mu\nu} &= u_\mu \nabla_\nu \left[(\varrho + p) u^\mu u^\nu + p g^{\mu\nu} \right] \\
&= u_\mu \left[\nabla_\nu (\varrho + p) u^\mu u^\nu + (\varrho + p)(\nabla_\nu u^\mu) u^\nu \right. \\
&\quad + \left. (\varrho + p) u^\mu \nabla_\nu u^\nu + (\nabla_\nu p) g^{\mu\nu} \right].
\end{aligned}
\tag{8.30}
$$

Let us now prove the following equality

$$
u_\mu \nabla_\nu u^\mu = 0.
\tag{8.31}
$$

Proof. Since $u^\mu u_\mu = -1$,, it is clear that $\nabla_\nu (u_\mu u^\mu) = 0$. This, in turn, gives

$$
\begin{aligned}
\nabla_\nu (u_\mu u^\mu) &= (\nabla_\nu u_\mu) u^\mu + u_\mu \nabla_\nu u^\mu = (\nabla_\nu u_\mu) u^\mu + u^\mu \nabla_\nu u_\mu \\
&= 2 (\nabla_\nu u_\mu) u^\mu = 0.
\end{aligned}
\tag{8.32}
$$

$\square$

Given the normalization of 4-velocity it follows from the equation (8.30) that

$$
\begin{aligned}
0 &= -\nabla_\nu (p + \varrho) u^\nu - (\varrho + p)\nabla_\nu u^\nu + u^\nu \nabla_\nu p \\
&= -u^\nu \nabla_\nu \varrho - (\varrho + p)\nabla_\nu u^\nu.
\end{aligned}
\tag{8.33}
$$

It can be shown that the particle density n in the given fluid is conserved, i.e.

$$
\nabla_\nu (n u^\nu) = 0,
\tag{8.34}
$$

which implies

$$
\nabla_\nu (n u^\nu) = (\nabla_\nu n) u^\nu + n \nabla_\nu u^\nu \quad \Rightarrow \quad \nabla_\nu u^\nu = -\frac{1}{n} u^\nu \nabla_\nu n = -\frac{1}{n} \frac{\mathrm{d}n}{\mathrm{d}t}.
\tag{8.35}
$$

Putting (8.35) into (8.33) we arrive at equation

$$
-\frac{\mathrm{d}\varrho}{\mathrm{d}t} + \frac{p + \varrho}{n} \frac{\mathrm{d}n}{\mathrm{d}t} = 0.
\tag{8.36}
$$

The particle density is inversely proportional to a proper volume $n \sim 1/V$ and, hence, $\mathrm{d}n/n = -\mathrm{d}V/V$. After substituting this into (8.36) we get

$$
\frac{\mathrm{d}\varrho}{\mathrm{d}t} + \frac{\varrho + p}{V} \frac{\mathrm{d}V}{\mathrm{d}t} = 0.
\tag{8.37}
$$

Multiplying this equation by volume V we obtain after some algebra

$$\frac{\mathrm{d}(\varrho V)}{\mathrm{d}t} + p\frac{\mathrm{d}V}{\mathrm{d}t} = 0. \tag{8.38}$$

This equation remains us the first law of thermodynamics for adiabatic process (as the right-hand side is zero). For the universe we speak of adiabatic expansion or compression.

Let us now calculate the volume of a universe V in a given time t:

$$V = \int \sqrt{^{(3)}g}\,\mathrm{d}\chi\mathrm{d}\theta\mathrm{d}\varphi = \int R^3(t)\Sigma_k^2(\chi)\sin\theta\mathrm{d}\chi\mathrm{d}\theta\mathrm{d}\varphi = 4\pi R^3\int \Sigma_k^2(\chi)\mathrm{d}\chi. \tag{8.39}$$

From this result, it is clear that $V \sim R^3$. After putting this into (8.38) we obtain the final form of the conservation law for the energy during adiabatic expansion (compression) of the universe:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\varrho R^3\right) + p\frac{\mathrm{d}R^3}{\mathrm{d}t} = 0.} \tag{8.40}$$

E5 Show that in FLRW cosmology ($\Lambda > 0$) all matter dominated universes with big bang as $t \to \infty$ asymptotically approach *de Sitter* universe and as $t \to 0$ they approach *Einstein-de-Sitter* universe.

▶ **Solution:** We can rewrite Friedmann's equation into the form

$$\dot{R}^2 = \frac{C}{R} - k + \frac{\Lambda}{3}R^2, \tag{8.41}$$

where $C = $ const. From this equation it is clear that for big R ($t \to \infty$) we have

$$\dot{R} \approx \sqrt{\Lambda/3}R \quad \to \quad R(t) \sim \exp(\sqrt{\Lambda/3}\,t), \tag{8.42}$$

which corresponds to *de-Sitter* universe ($k = 0$, $\varrho = 0$, $\Lambda > 0$), whereas for small R ($t \to 0$) we get

$$\dot{R} \approx C/\sqrt{R} \quad \to \quad R(t) \sim t^{2/3}, \tag{8.43}$$

which corresponds to *Einstein-de-Sitter* universe ($k = 0$, $p = 0$, $\Lambda = 0$).

E6 One of the physical interpretations of cosmological constant is through the notion of scalar field, φ, called the "quitessence". Assuming spatial

homogenity ($\nabla_i \varphi = 0$) of the field and R-W metric, the equations of motion of the scalar field φ read

$$\ddot{\varphi} + 3\,H\,\dot{\varphi} + \frac{\mathrm{d}V}{\mathrm{d}\varphi} = 0. \tag{8.44}$$

where is $V = V(\varphi)$ potential of the scalar field and H is the Hubble parameter. Show that for sufficiently large value of H and for $\nabla_\mu \varphi <<V(\varphi)$ we will obtain stress-energy tensor for a cosmological constant.

▶ **Solution:** The stress-energy tensor of scalar field reads

$$T_{\mu\nu} = \nabla_\mu \varphi \, \nabla_\nu \varphi + g_{\mu\nu} \left(\frac{1}{2} g^{\alpha\beta} \nabla_\alpha \varphi \, \nabla_\beta \varphi - V(\varphi) \right). \tag{8.45}$$

The homogenity requires the validity of the condition

$$\nabla_i \varphi = 0 \tag{8.46}$$

where i runs over spatial indexes. The non-zero components of stress-energy tensot $T_{\mu\nu}$ are

$$T_{TT} = \frac{1}{2}\dot{\varphi}^2 + V(\varphi), \tag{8.47}$$

$$T_{ij} = -g_{ij}V(\varphi). \tag{8.48}$$

The condition $\dot{\varphi} << V(\varphi)$ further simplifies the TT component of the stress-energy tensor, which now reads

$$T_{TT} \simeq V(\varphi). \tag{8.49}$$

Provided that potential $V(\varphi)$ has an interval where it is almost constant in field φ we find out that the cosmological constant stress-energy tensor can be written in terms of scalar field φ. Corresponding energ-density ϱ_Λ and pressure p_Λ read

$$\varrho_\Lambda \simeq V(\varphi) \tag{8.50}$$

and

$$p_\Lambda \simeq -V(\varphi). \tag{8.51}$$

Recall that equation of state of cosmological constant reads $p_\Lambda = -\varrho_\Lambda$.

E7* The de Sitter in the (t, x, y, z) coordinates reads

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \exp 2H\,t \left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right). \qquad (8.52)$$

In this spacetime, let's assume non-comoving observer and solve the geodesic equation for them. Find the affine parameter as function of cosmic time t and show that the geodesic will reach $t \to -\infty$ in a finite affine parameter. This example shows, that selected coordinates do not cover the whole manifold.

E8* The non-relativistic particles have zero temperature T and zero pressure p. This is a cosmological idealization, however in reality those particles posses some temperature and pressure, due to their random motion. They satisfy the equation

$$p \sim \varrho\, T. \qquad (8.53)$$

(A) Find out what is the dependence of the gas of massive particles on the scale parameter.

(B) Let's consider neutrinos with mass $m_\nu = 0.1\mathrm{eV}$, and a present epoch temperature $T_{\nu 0} = 2\mathrm{K}$. Find out the redshift corresponding to the moment when neutrinos become non-relativistic.

E9 Determine comoving particle horizon $\Delta\chi_*$ and physical particle horizon d_* sizes at given epoch identified with scale factor value a_*, assuming the universe is filled with:

(A) dust $(p = 0)$,
(B) radiation $(p = \varrho/3)$.

Compare those comoving horizons with comoving distance χ_{AB} separating a point A on CMBR from point B on Earth.

▶ **Solution:** Without loss of generality, assume that the photon follows radial null geodesics, described with line element

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\mathrm{d}\chi^2. \qquad (8.54)$$

Clearly, the comoving distance $\Delta\chi$ is given by formula

$$\Delta\chi = \int_{t_1}^{t_2} \frac{\mathrm{d}t}{a(t)} = \int_{a_1}^{a_2} \frac{\mathrm{d}a}{\dot{a}\,a} \quad (\text{ Why?}) \qquad (8.55)$$

The evolution of scale parameter $a(t)$ with cosmic time t is determined by the fluid properties. For presureless dust the equation of state reads

$$p = 0. \tag{8.56}$$

Let's recall that the conservation law of the energy reads

$$\mathrm{d}\left(\varrho a^3\right) + p \mathrm{d}a^3 = 0. \tag{8.57}$$

In the case of dust filled universe the last equation simplifies to one that reads

$$\mathrm{d}\left(\varrho a^3\right) = 0 \;\Rightarrow\; \varrho(t)\, a^3(t) = \text{const.} \tag{8.58}$$

Normalizing the scale factor at present epoch to $a_0 = 1$ the Friedman equation will now read

$$H^2 = \frac{8\pi\, G}{3}\frac{\varrho_0}{a^3} = \frac{H_0^2}{a^3} \;\Rightarrow\; \dot{a} = H_0\, a^{-1/2}. \tag{8.59}$$

Using last equation in (8.55) we obtain formula for comoving distance of radiation in case of dust filled universe in the form

$$\Delta\chi = \int_{a_1}^{a_2}\frac{\mathrm{d}a}{H_0 a^{1/2}} = \frac{2}{H_0}\left(a_2^{1/2} - a_1^{1/2}\right). \tag{8.60}$$

The comoving particle horizon distance at epoch t_* is the distance that a photon travels from the Big Bang to the epoch with corresponding scale parameter a_*, i.e.

$$\Delta\chi_* = \frac{2}{H_0}a_*^{1/2}. \tag{8.61}$$

Corresponding physical particle horizon distance reads

$$\Delta d_* = a_*\,\Delta\chi_* = \frac{2}{H_*}. \tag{8.62}$$

In the case of radiation the equation of state reads

$$p = \frac{1}{3}\varrho \tag{8.63}$$

and the conservation law (8.57) will give equation

$$\frac{\mathrm{d}\varrho}{\varrho} = -4\frac{\mathrm{d}a}{a} \;\Rightarrow\; \varrho(t)a^4(t) = \text{const.} \tag{8.64}$$

Following similar procedure as in case of dust filled universe we will arrive to formulas for particle horizon distances in the form

$$\Delta\chi_* \;=\; \frac{a_*}{H_0}, \tag{8.65}$$

$$d_* \;=\; a_*\Delta\chi_* = \frac{a_*^2}{H_0} = \frac{1}{H_*}. \tag{8.66}$$

Looking back at CMBR we observe the Universe when its scale factor was $a_{\mathrm{CMBR}} \sim 1/1300$. The comoving distance between a point A at last scattering surface and point B on Earth now is in case of dust filled universe

$$\chi_{AB} = \frac{2}{H_0}\left(1 - a_{\mathrm{CMBR}}^{1/2}\right) \sim \frac{2}{H_0}. \tag{8.67}$$

The particle horizon distance for point A is

$$\Delta\chi_*(A) = \frac{2}{H_0}a_{\mathrm{CMBR}}^{1/2}. \tag{8.68}$$

Causally connected region around point A is much smaller then the comoving distance between A and B. Problem is that CMBR is highly isotropic even for causally disconnected regions. We call this "horizon problem" of standard model.

E10 Explain the nature of "Flatness problem" and show how inflation theory resolves this problem

▶ **Solution:** Let us start with Friedman equation

$$\dot{H}^2 = \frac{8\pi\,G}{3}(\varrho_M + \varrho_\gamma) - \frac{k}{a^2}. \tag{8.69}$$

Using dimensionless density parameters

$$\Omega_i = \frac{\varrho_i}{\varrho_c}, \tag{8.70}$$

where is ϱ_c critical density for spatially flat universe, i.e.

$$\varrho_c = \frac{3H^2}{8\pi G}, \tag{8.71}$$

we rewrite Friedman equation to read

$$1 = \Omega_m + \Omega_\gamma + \Omega_K, \tag{8.72}$$

where we defined

$$\Omega_K = -\frac{k}{H^2 a^2} = -\frac{k}{\dot{a}^2}. \tag{8.73}$$

From CMBR observations it follows that $\Omega_K = 0$ best fits the data. Ω_K is clearly time dependent parameter. In the regime of dust domination , i.e. from the epoch when Universe has cooled to 10^4K, to present epoch the scale parameter evolved with time according to relation $a(t) \sim t^{2/3}$ and therefore the Ω_K parameter scales as $\Omega_K \sim T^{-1}$ (clearly, we have $\Omega_K \sim \dot{a}^{-2} \sim t^{2/3}$). Since there is $a \sim T^{-1}$ then also $\Omega_K \sim T^{-1}$. From the observations we can safely consider that $|\Omega_K| < 1$. In order to satisfy this condition at the epoch when cosmic soup was 10^4K hot then $|\Omega_K| < 10^{-4}$. When temperature of the cosmic content increases above 10^4K then its evolution is driven by radiation domination regime, there $a \sim t^{1/2}$. In this case the curvature parameter scales as $\Omega_K(t) \sim a^2(t) \sim T^{-2}$. If Ω_K be 10^{-4} at the radiation dominated epoch at temperature $T \sim 10^{10}$K (electron-positron anihilation era) then there is $|\Omega_K| < 10^{-16}$ (Why?). In order to obtain present curvature parameter $|\Omega_K| < 1$ it must be finely tuned to zero at early stages of the Universe evolution.

Inflation theory introduces a new phase taking place just Plank time after Big Bang and that last just 10^{-35}s, but during that stage it undergoes rapid expansion due to inflation field with equation of state $p = -\varrho = const$. This is the period where Hubble parameter H, remains constant. The Friedman equation reads

$$H^2 = \frac{8\pi G}{3} \varrho_{vac} \tag{8.74}$$

and therefore scale parameter evolves exponentially, i.e.

$$a(t) = a_{Pl} \exp\left[\sqrt{\frac{8\pi G}{3} \varrho_{vac}}(t - t_{Pl})\right]. \tag{8.75}$$

The Friedman equations, including curvature term, during inflation period reads

$$H^2 = \frac{8\pi G}{3} \varrho_{vac} - \frac{k}{a^2}. \tag{8.76}$$

Since ϱ_{vac} and H are constant then sufficiently large inflation will render Ω_k sufficiently small so we can now observer its value almost zero, for any initial curvature.

E11 Let p and ϱ are comoving pressure and energy density of the perfect fluid filling up the model of the Universe. Show that equation of state

$$p = -\varrho \tag{8.77}$$

implies that $\varrho = \varrho_0 = const.$

▶ **Solution:** The energy-momentum conservation equation in the case of expanding universe reads

$$\mathrm{d}\left(\varrho\, a^3\right) + p\mathrm{d}a^3 = 0. \tag{8.78}$$

Using equation of state (8.77) then leads to equation (show)

$$a^3\mathrm{d}\varrho = 0 \tag{8.79}$$

which implies result (since scale factor is clearly non zero)

$$\mathrm{d}\varrho = 0 \;\Rightarrow\; \varrho = \varrho_0 = const. \tag{8.80}$$

E12 Assume the initial inflation phase of evolution of the Universe and let the inflation field be cosmological constant $\Lambda = 8\pi\varrho_{vac} > 0$ with corresponding equation of state $p = -\varrho$. Determine the temporal evolution of the scale factor a during this phase of the Universe evolution.

▶ **Solution:** The Friedman equation reads

$$H^2 = \frac{8\pi\,G}{3}\varrho_{vac} - \frac{k}{a^2}. \tag{8.81}$$

As shown in the previous example $\varrho_{vac} = const.$, then the temporal evolution of $a(t)$ is simply encoded in the integral

$$t = \int \left(\frac{8\pi\,G\,\varrho_{vac}}{3}a^2 - k\right)^{-1/2}\mathrm{d}a. \tag{8.82}$$

For three different curvature parameters $k = -1, 0, 1$ we obtain result

$$a(t) \sim \begin{cases} \cosh^{-1}(H\,t) & \text{for} \quad k = -1, \\ \exp(H\,t) & \text{for} \quad k = 0, \\ \sinh^{-1}(H,\,t) & \text{for} \quad k = +1. \end{cases} \tag{8.83}$$

Here parameter $H = \sqrt{8\pi\,G\,\varrho_{vac}/3}$. This is identical to Hubble parameter only in case of $k = 0$. It is worth to note, that in both cases $k = \pm 1$ the expansion approaches exponential expansion of case $k = 0$. It means, that for large enough inflation phase the expansion parameter will grow the same regardless initial curvature parameter k.

9
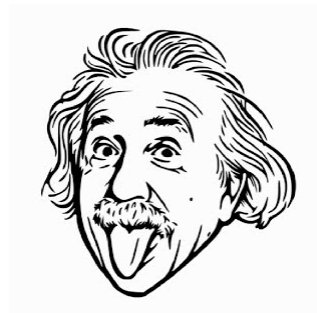
Einstein's Equations

Finally, a section devoted to the Einstein's field equations themselves. The field equations tell how the energy-mass distribution generates curved geometry of the world and how the geometry influences the energy-mass distribution. In this section we will practise the derivation of Einstein's equations [3, 6, 7, 9].

E1 Show that the vacuum Einstein's field equations follow from the Einstein-Hilbert action

$$S = \int \sqrt{-g} R \, \mathrm{d}^4 x \qquad (9.1)$$

where R is Ricci scalar and g is the determinat of the metric $g_{\mu\nu}$ (hint: use identity $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$).

▶ **Solution:** The Einstein's field equations come from the variational principle of least action which states

$$\delta S = 0 \,, \qquad (9.2)$$

where the operator δ is a variation. Applying it to (9.1) one obtains

$$
\begin{aligned}
\delta S &= \int \delta\left(\sqrt{-g}R\right) = \int \left(R\delta\sqrt{-g} + \delta R\sqrt{-g}\right)\mathrm{d}^4 x \\
&= \int \left(R\delta\sqrt{-g} + \sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu} + \sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}\right)\mathrm{d}^4 x. \quad (9.3)
\end{aligned}
$$

The variation of the Ricci tensor is known to be of the form

$$\int \sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}\mathrm{d}^4 x = \int \sqrt{-g}\nabla_\mu V^\mu \mathrm{d}^4 x, \qquad (9.4)$$

for certain V^μ. Notice, however, that the quantity $\sqrt{-g}\nabla_\mu V^\mu$ represents a total derivative of V^μ and, therefore, from Gauss theorem, one finds out that

$$\int_\Omega \sqrt{-g}\nabla_\mu V^\mu \mathrm{d}^4 x = \int_{\partial\Omega} \mathrm{d}V^\mu = 0. \tag{9.5}$$

Equation (9.3) now reads

$$\begin{aligned}
\delta S &= \int \left(R\delta\sqrt{-g} + \sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu}\right)\mathrm{d}^4 x \\
&= \int \sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu}\mathrm{d}^4 x = 0.
\end{aligned} \tag{9.6}$$

In the last step, the hint was used. The last equation must hold for any $\delta g^{\mu\nu}$ and therefore we obtain the desired result

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0. \tag{9.7}$$

E2 Derive Einstein's field equations via *Palatini* variation. In other words, assume no a-priori relationship between the affine connection $\Gamma^\varrho_{\mu\nu}$ and the metric $g_{\mu\nu}$ and use the action principle independently for both.

▶ **Solution:** Variation of the Einstein-Hilbert action with respect to the (inverse) metric reads
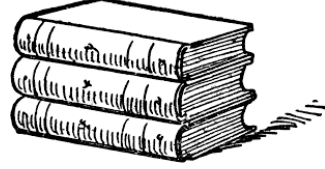
$$\frac{\delta S}{\delta g^{\mu\nu}} = 0 \quad \Rightarrow \quad \frac{\delta\left(\sqrt{-g}g^{\alpha\beta}\right)}{\delta g^{\mu\nu}}R_{\alpha\beta} = \sqrt{-g}\left(R^{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = 0. \tag{9.8}$$

These would be Einstein's equations if the connections inside $R_{\mu\nu}$ and R were the same as Christoffel symbols. That is indeed obtained once we vary the action with respect to the connection, which yields

$$\frac{\delta S}{\delta\Gamma^\varrho_{\mu\nu}} = 0 \quad \Rightarrow \quad \partial_\varrho g^{\mu\nu} = \Gamma^\mu_{\varrho\sigma}g^{\sigma\nu} + \Gamma^\nu_{\varrho\sigma}g^{\mu\sigma}. \tag{9.9}$$

This is nothing but the metric compatibility condition $\nabla_\varrho g^{\mu\nu} = 0$.

10

Misc problems

Here, various problems of relativistic physics for gaining deeper insight into relativistic physics are presented [4, 8, 5].

E1 Consider the motion of a neutral test particle in the Reissner-Nordström (R-N) metric field. Its spacetime interval reads

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + f(r)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right) \tag{10.1}$$

where

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \tag{10.2}$$

with Q being an electric charge of the R-N spacetime. Show that there exists a radius $r = r_{\min}$ where a neutral test particle can remain still (the boundary of the repulsive region).

▶ **Solution:** Let us assume motion in the equatorial plane. Then radial component of the geodesic equation reads

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = E^2 - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)\left(1 + \frac{L^2}{r^2}\right). \tag{10.3}$$

Consider a particle with zero angular momentum $L = 0$.

$$V_{\mathrm{eff}} \xrightarrow[L=0]{} 1 - \frac{2M}{r} + \frac{Q^2}{r^2} = f(r). \tag{10.4}$$

If there is a stable boundary $r = r_{\min}$ between the repulsive and attractive character of the geometry, then there the following conditions must be fulfilled: $\left. \mathrm{d}V_{\mathrm{eff}}/\mathrm{d}r \right|_{r_{\min}} = 0$ and $\left. \mathrm{d}^2 V_{\mathrm{eff}}/\mathrm{d}r^2 \right|_{r_{\min}} > 0$.

Let us do the calculations:

$$\left. \frac{\mathrm{d}f(r)}{\mathrm{d}r} \right|_{r_{\min}} = \frac{2M}{r_{\min}^2} - \frac{2Q^2}{r_{\min}^3} = 0 \qquad \Rightarrow \qquad \boxed{r_{\min} = \frac{Q^2}{M}.} \quad (10.5)$$

$$\Downarrow$$

$$\left. \frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2} \right|_{r_{\min}} = -\frac{4M}{r_{\min}^3} + \frac{6Q^2}{r_{\min}^4} = \frac{2M^3}{Q^6} > 0. \quad (10.6)$$

E2 In the field of R-N naked singularity, there are two test particles sent toward its center. Both particles have covariant energy $E = 1$, angular momentum $L = 0$ and rest mass m. The particles are sent with a given mutual time-delay. After the first particle reaches the turning point, it turns back and collides with the incoming second particle at some r. Determine the energy of the collision. Determine the location of the collision with has maximal energy.

▶ **Solution:** The R-N geometry is given by the equation (10.1). The radial component of particle's 4-velocity reads

$$U^t = g^{tt}U_t = -\frac{U_t}{f} = \frac{E}{f} = \frac{1}{f}, \quad (10.7)$$

$$\left(U^r\right)^2 = E^2 - V_{eff} = E^2 - f = 1 - f \qquad \Rightarrow \qquad U^r = \pm\sqrt{1 - f}. \quad (10.8)$$

In the moment of the collision, the 4-velocities of ingoing and outgoing particles are given by formulae

$$U_1 = (1/f, \sqrt{1 - f}, 0, 0), \qquad U_2 = (1/f, -\sqrt{1 - f}, 0, 0). \quad (10.9)$$

The energy of collision reads

$$E_{CM}^2 = -(P_1 + P_2)^2 = m_1^2 + m_2^2 - 2g_{\mu\nu}P^\mu P^\nu = 2m^2 - 2m^2 g_{\mu\nu}U^\mu U^\nu =$$
$$= 2m^2\left[1 - \left(-\frac{f}{f^2} - \frac{1}{f} + 1\right)\right] = \frac{4m^2}{f}. \quad (10.10)$$

The maximal energy of the collision $E_{CM,\,\text{max}}^2$ will be reached at the minima of the function $f(r)$, i.e. at the point $r_{\text{min}} = Q^2/M$. The formula for maximal energy of the collision then reads

$$E_{CM,\,\text{max}}^2 = \frac{4m^2}{1 - \frac{M^2}{Q^2}}\,. \tag{10.11}$$

E3 Find out the amount of local acceleration, necessary to keep the test particle at rest at a given radius r. Determine the magnitude of this acceleration at the event horizon.

▶ **Solution:** Consider spherically symmetric spacetime with spacetime interval

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + f^{-1}(r)\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2\,. \tag{10.12}$$

The 4-acceleration of particle having 4-velocity u^α is given by the equation

$$a^\alpha \equiv \frac{\mathrm{d}u^\alpha}{\mathrm{d}\tau} = u^\alpha{}_{;\beta}u^\beta = u^\alpha{}_{,\beta}u^\beta + \Gamma^\alpha_{\beta\gamma}u^\beta u^\gamma\,. \tag{10.13}$$

The components of stationary particle 4-velocity are $u^\mu = (u^t, 0, 0, 0)$. The normalization condition implies $u^t = 1/\sqrt{-f(r)}$. Thus, we have

$$a^\alpha = \Gamma^\alpha_{tt}(u^t)^2 = -\Gamma^\alpha_{tt}\frac{1}{f(r)}\,. \tag{10.14}$$

Clearly, $\Gamma^t_{tt} = \Gamma^\theta_{tt} = \Gamma^\varphi_{tt} = 0$ and

$$\Gamma^r_{tt} = \frac{1}{2}g^{rr}\left(-g_{tt,r}\right) = \frac{1}{2}ff'\,. \tag{10.15}$$

The only non-zero component of 4-acceleration is the radial one, i.e. $a^r = -f'/2$. The magnitude of the 4-acceleration thus reads

$$a(r) = \sqrt{g_{\alpha\beta}a^\alpha a^\beta} = \sqrt{g_{rr}(a^r)^2} = \frac{f'}{2\sqrt{f}}\,. \tag{10.16}$$

Since $f = 0$ at the event horizon we get the expected result $a(r_{hor}) = \infty$.

E4 Consider a test particle which is lowered by a distant observer (usually at infinity) toward the radius r using infinitely long, massless string. What

is the magnitude of $a_\infty(r)$, of the observer's action on test particle? What is the magnitude of $a_\infty(r_{hor})$?

▶ **Solution:** Consider following mind-experiment. Observer at infinity lowers the string by a small proper length δs. Doing so, he performs work

$$\delta W_\infty = a_\infty \delta s \,. \tag{10.17}$$

At radius r, the particle moves by proper distance δs, but performed work is

$$\delta W = a(r) \delta s \,. \tag{10.18}$$

Now, let the work δW be turned to radiation, which is collected at infinity. Energy of this radiation suffers from gravitational redshift

$$\delta E_\infty = f^{1/2} \delta \omega = f^{1/2} a \delta s \,. \tag{10.19}$$

The energy conservation implies $\delta E_\infty = \delta W_\infty$ and thus

$$f^{1/2} a \delta s = a_\infty \delta s \,, \quad \Rightarrow \quad a_\infty = f^{1/2} a \,. \tag{10.20}$$

Inserting into this the expression for a from Eq. (10.16) we will get

$$\boxed{a_\infty = \frac{1}{2} f' \,.} \tag{10.21}$$

The magnitude of the acceleration $a_\infty(r_{hor})$ is finite at the event horizon.

E5 The Lagrange density of the complex scalar field φ with a mass μ reads

$$\mathcal{L} = -g^{ij} \nabla_i \varphi \nabla_j \varphi^* - \mu^2 \varphi \varphi^* \,. \tag{10.22}$$

Using this Lagrange density determine equations of motion of the complex scalar field φ.

▶ **Solution:** Corresponding equations of motion follow from Euler-Lagrange equations. For the field φ^* they read

$$\nabla_i \left(\frac{\partial \mathcal{L}}{\partial (\nabla_i \varphi^*)} \right) - \frac{\partial \mathcal{L}}{\partial \varphi^*} = 0 \tag{10.23}$$

At first, we determine the quantity

$$\frac{\partial \mathcal{L}}{\partial (\nabla_i \varphi^*)} = -g^{ij} \nabla_j \varphi \tag{10.24}$$

and

$$\frac{\partial \mathcal{L}}{\partial \varphi^*} = -\mu^2 \varphi. \tag{10.25}$$

Inserting the last two results into Euler-Lagrange equations (10.23) we obtain

$$\begin{aligned}
\nabla_i \left(-g^{ij}\nabla_j\varphi\right) + \mu^2\varphi &= 0 \\
-\underbrace{\nabla_i g^{ij}}_{=0}\nabla_j\varphi - g^{ij}\nabla_i\nabla_j\varphi + \mu^2\varphi &= 0 \\
\left(\nabla^i\nabla_i - \mu^2\right)\varphi &= 0 \\
\left(\Box - \mu^2\right)\varphi &= 0.
\end{aligned} \tag{10.26}$$

E6 Show that the d'Alambert operator $\Box$ for scalar function φ can be written in the form

$$\Box\varphi = \left(\partial^i\partial_i - \Gamma^i{}_{ik}\partial^k\right)\varphi. \tag{10.27}$$

▶ **Solution:** The validity of the equation (10.27) can be checked by direct calculation:

$$\begin{aligned}
\Box\varphi &= \nabla_i\nabla^i\varphi = \nabla_i\left(g^{is}\nabla_s\varphi\right) = |\nabla_i g^{is} = 0| = g^{is}\nabla_i\nabla_s\varphi \\
&= |\nabla_s\varphi = \partial_s\varphi| = g^{is}\nabla_i(\partial_s\varphi) = \nabla_i(\partial^i\varphi) = \partial_i\partial^i\varphi + \Gamma^i{}_{is}\partial^s\varphi \\
&= \left(\partial_i\partial^i + \Gamma^i{}_{is}\partial^s\right)\varphi. \tag{10.28}
\end{aligned}$$

E7 Using the fomula

$$\Gamma^i{}_{ij} = \frac{1}{\sqrt{-g}}\partial_j\sqrt{-g}, \tag{10.29}$$

where is g the determinant of the metric g_{ij}, rewrite the Klein-Gordon equation as

$$\left[\frac{1}{\sqrt{-g}}\partial_j\left(g^{jk}\sqrt{-g}\partial_k\right) - \mu^2\right]\varphi = 0. \tag{10.30}$$

▶ **Solution:** We insert the relation (10.29) into (10.27) to obtain

$$\left(\partial_i \partial^i + \Gamma^i{}_{ik}\partial^k - \mu^2\right)\varphi = 0$$

$$\left(\partial_i \partial^i + \frac{1}{\sqrt{-g}}\partial_k \sqrt{-g}\partial^k - \mu^2\right)\varphi = 0$$

$$\left[\frac{1}{\sqrt{-g}}\left(\sqrt{-g}\partial_i\partial^i + \partial_k\sqrt{-g}\partial^k\right) - \mu^2\right]\varphi = 0$$

$$\left[\frac{1}{\sqrt{-g}}\partial_i\left(\sqrt{-g}\partial^i\right) - \mu^2\right]\varphi = 0. \qquad (10.31)$$

$\boxed{\text{E8}}$ Find equations of motion for a scalar field in the Schwarzschild black hole background. Look for a solution in separated form.

▶ **Solution:** Non-zero covariant components of the metric tensor for Schwarzschild spacetime read

$$g_{tt} = -\left(1 - \frac{2}{r}\right),\ g_{rr} = \left(1 - \frac{2}{r}\right)^{-1},\ g_{\theta\theta} = r^2,\ g_{\varphi\varphi} = r^2\sin^2\theta. \quad (10.32)$$

The determinant of the metric is given by

$$g = -r^4\sin^2\theta. \qquad (10.33)$$

We now express the equation (10.31) in terms of Schwarzschild metric and we obtain

$$\left\{\frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial t}\left(g^{tt}r^2\sin\theta\frac{\partial}{\partial t}\right) + \frac{\partial}{\partial r}\left(g^{rr}r^2\sin\theta\frac{\partial}{\partial r}\right) + \right.\right.$$

$$\left.\left. +\frac{\partial}{\partial\theta}\left(g^{\theta\theta}r^2\sin^2\theta\frac{\partial}{\partial\theta}\right) + \frac{\partial}{\partial\varphi}\left(g^{\varphi\varphi}r^2\sin\theta\frac{\partial}{\partial\varphi}\right)\right] - \mu^2\right\}\Phi = 0. \quad (10.34)$$

After some algebra we have the final form

$$-\frac{r^3}{r-2}\frac{\partial^2\Phi}{\partial t^2} + \frac{1}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2} + \frac{\partial}{\partial r}\left(r(r-2)\frac{\partial\Phi}{\partial r}\right) +$$

$$+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta} - \mu^2 r^2\Phi\right) = 0. \quad (10.35)$$

Due to spherical symmetry and temporal independence of Schwarzschild metric we seek the solution of this differential equation in the form

$$\Phi = e^{-i\omega t}R(r)A(\theta,\varphi). \qquad (10.36)$$

Now, let us insert (10.36) into (10.35) and calculate individual partial derivatives first. We obtain

$$\frac{\partial \Phi}{\partial t} = -i\omega e^{-i\omega t}RA, \qquad \frac{\partial^2 \Phi}{\partial t^2} = -\omega^2 e^{-i\omega t}RA, \qquad (10.37)$$

$$\frac{\partial \Phi}{\partial \varphi} = e^{-i\omega t}R\frac{\partial A}{\partial \varphi}, \qquad \frac{\partial^2 \Phi}{\partial \varphi^2} = e^{-i\omega t}R\frac{\partial^2 A}{\partial \varphi^2}, \qquad (10.38)$$

and

$$\frac{\partial}{\partial r}\left((r^2 - 2r)\frac{\partial \Phi}{\partial r}\right) = 2(r-1)e^{-i\omega t}A\frac{dR}{dr} + r(r-2)e^{-i\omega t}A\frac{d^2 R}{dr^2}, \quad (10.39)$$

$$\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \Phi}{\partial \theta}\right) = \cos\theta e^{-i\omega t}R\frac{\partial A}{\partial \theta} + \sin\theta e^{-i\omega t}R\frac{\partial^2 A}{\partial \theta^2}. \qquad (10.40)$$

Thus, we have arrived at the equation

$$\frac{r^3}{r-2}\omega^2 + \frac{1}{A\sin^2\theta}\frac{\partial^2 A}{\partial \varphi^2} + \frac{2(r-1)}{R}\frac{dR}{dr}$$
$$+\frac{r(r-2)}{R}\frac{d^2 R}{dr^2} + \frac{\cot\theta}{R}\frac{\partial A}{\partial \theta} + \frac{1}{A}\frac{\partial^2 A}{\partial \theta^2} - \mu^2 r^2 = 0, \qquad (10.41)$$

This equation is clearly separable into radial and angular parts, i.e. we write down this separation in symbolic way

$$P(r) = W(\theta, \varphi). \qquad (10.42)$$

Clearly, this equation must be fullfilled for any r, θ and φ, i.e. both, left and right sides must be equal to the same constant, say K. In this way we get two equations

$$r(r-2)\frac{d^2 R}{dr^2} + 2(r-1)\frac{dR}{dr} + \left(\frac{r^3\omega^2}{r-2} - \mu^2 r^2 - K\right)R = 0 \qquad (10.43)$$

and

$$\frac{1}{\sin^2\theta}\frac{\partial^2 A}{\partial \varphi^2} + \cot\theta\frac{\partial A}{\partial \theta} + \frac{\partial^2 A}{\partial \theta^2} - KA = 0. \qquad (10.44)$$

E9* Assume a braneworld static black hole solution with the spacetime interval

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \qquad (10.45)$$

where

$$f(r) = \left(1 - \frac{2M}{r} + \frac{b}{r^2}\right). \qquad (10.46)$$

(A) Discuss existence/non-existence of the event horizon with respect to the tidal charge parameter value b.

(B) Determine loci of the marginally stable and photon orbits.

(C) Determine Keplerian frequency $\Omega_K \equiv u^\varphi/u^t$ (u^μ is 4-velocity vector) of test massive particle on circular orbit and discuss its properties with respect to braneworld tidal charge parameter b.

(D) Determine frequency shift g of the radiation emitted from the source on circular orbit and discuss its properties with respect to the tidal charge parameter. Recall that g is defined as

$$g \equiv \frac{k_\mu u^\mu|_{ob}}{k_\mu u^\mu|_{em}} \tag{10.47}$$

where is k^μ the photon propagation $4-$vector, u^μ is the 4-velocity of observer (ob)/emitter (em).

E10* Consider a scalar field φ of mass m determined by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\left(\nabla\varphi\right)^2 + \frac{1}{2}m^2\varphi^2. \tag{10.48}$$

Discuss the violation of the following:

(A) *Strong Energy Condition* which states

$$\left(T_{\mu\nu} - \frac{T}{2}g_{\mu\nu}\right)V^\mu V^\nu \geq 0 \tag{10.49}$$

where $T_{\mu\nu}$ are components of stress-energy tensor $T = T^\mu{}_\mu$ and V^μ is any time-like 4-vector.

(B) *Null Energy Condition* stating

$$T_{\mu\nu}k^\mu k^\nu \geq 0 \tag{10.50}$$

where is k^μ any null 4-vector.

(C) *Weak Energy Condition* that states

$$T_{\mu\nu}V^\mu V^\nu \geq 0. \tag{10.51}$$

Recall that the stress-energy tensor corresponding to scalar field φ reads

$$T^{\mu\nu} = \nabla^\mu\varphi\,\nabla^\nu\varphi - \frac{1}{2}g^{\mu\nu}\left((\nabla\varphi)^2 + m^2\varphi^2\right). \tag{10.52}$$

E11* Consider Kehagios-Sfetsos (K-S) compact object. Its geometry is given by the spacetime interval

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{1}{f(r)}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2) \qquad (10.53)$$

where is

$$f(r) = 1 + r^2\omega\left[1 - \left(1 + \frac{4M}{\omega r^3}\right)^{1/2}\right]. \qquad (10.54)$$

Analyze the circular photon and massive particle orbits to show that

(A) there are two photon orbits for Hořava parameter $\omega > \omega_{ms.ph} = \frac{2}{3\sqrt{3}}$ (for $\omega < \omega_{ms.ph}$ there are no photon circular orbits - why?)
(B) there is particular interval of Hořava parameter $\omega_{ms} < \omega < \omega_{ms.ph}$ where there exist two marginally stable orbits of massive particles.
(C) there is a static radius r_{stat} in K-S spacetime, where a massive test particle remains still relative to static observers at infinity and is given by formula
$$r_{stat}(\omega) = (2\omega)^{-1/3}. \qquad (10.55)$$

11

Newton's epilogue

 It is good to remaind that Newton's theory of gravity still holds in the regime of small velocities and weak gravitational field. Let us in few problems recall the way Newton's theory make it possible to touch planets and stars. The key ingredients are the law of gravitational force

$$\vec{F} = -G\frac{m\,M}{r^3}\vec{r} \tag{11.1}$$

stating that the gravitational force is colinear with radius vector r but is of opposite direction and its magnitude if proportional to masses m and M that are mutually attracted and that the force magnitude decays with square root the mutual distance of the bodies.

$\boxed{\text{E1}}$ Determine the gravitational constant G with torque pendulum.

▶ **Solution:** This method, to dermine gravitational constant, was performed by sir Cavendish in the period 1797-1798. Let us consider a device depicted in Fig. (11.1). At the end of the wire, there is a rod with two balls of masses m attached to its endpoints. Two masses M attract the smaller masses m due to mutual gravitational force. As the wire is twisted then its tension generates torque τ' against to the torque τ of the rod. When angle $\alpha = \alpha_0$ the balance between the two torques is established. Let us write down the mathematics behind this experiment.

1. The gravitational attraction between bodies m and M is determined by Newton's law of gravity (11.2). The magnitude of gravitational force between two bodies with masses m and M separated by the distance d

reads

$$F = G\frac{m\,M}{d^2}. \tag{11.2}$$

2. Both small bodies act on the wire with the torque

$$\tau = \frac{L}{2}F + \frac{L}{2}F = L\,F. \tag{11.3}$$

3. The balance between the wire torque τ' and the gravity force torque τ is established at angle α_0. For small α_0, Hooks law applies, i.e.

$$\tau' = -\kappa\,\alpha_0 \tag{11.4}$$

and the balance leads to the equation

$$|\tau'| = |\tau| \;\Rightarrow\; \kappa\,\alpha_0 = L\,F \tag{11.5}$$

or, when using equation (11.2) for the force F, to the equation

$$\kappa\,\alpha_0 = L\,G\,\frac{m\,M}{d^2}. \tag{11.6}$$

In order to determine the Gravitational constant the wire tension parameter κ must be determined. Let us remove, for a while, large masses M from the experiment. We are left with the torque pendulum. The torque τ and angular momentum B are related by the equation

$$\frac{\mathrm{d}\vec{B}}{\mathrm{d}t} = \vec{\tau} \tag{11.7}$$

where angular momentum $\vec{B}$ is given by formula

$$\vec{B} = I\,\vec{\omega} \tag{11.8}$$

where I is moment of inertia od pendulum and $\vec{\omega}$ is its angular frequency. In therms of norms of $\vec{\tau}$ and $\vec{B}$ the equation of motion of torque pendulum reads

$$I\,\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\kappa\,\alpha \tag{11.9}$$

or better

$$\frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2} = -\frac{\kappa}{I}\,\alpha. \tag{11.10}$$

Clearly, the solution to this equation will read

$$\alpha(t) = \alpha_0 \cos \sqrt{\frac{\kappa}{I}} t \tag{11.11}$$

assuming that the initial conditions are $\alpha(t = 0) = \alpha_0$ and $\dot{\alpha}(t = 0) = 0$. The corresponding angular frequency, clearly, is given by the formula

$$\omega = \sqrt{\frac{\kappa}{I}}. \tag{11.12}$$

The period of this pendulum is easily determined to read

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{I}{\kappa}}. \tag{11.13}$$

For the case of the rod with two same balls of masses m attached to its ends, the moment of inertia reads

$$I = m \left(\frac{L}{2}\right)^2 + m \left(\frac{L}{2}\right)^2 = \frac{1}{2} m L^2. \tag{11.14}$$

The period of the torque pendulum and the coefficient κ are related by the formula

$$\kappa = \frac{2\pi^2 \, m \, L^2}{T^2}. \tag{11.15}$$

Finally putting (11.15) to (11.6) we arrive to formula, that determines the value of the gravitational constant G in the form

$$G = \frac{2\pi^2 \alpha_0 d^2 \, L}{M \, T^2}. \tag{11.16}$$

$\boxed{\text{E2}}$ Using knowledge of the gravitational constant magnitude, determine mass of the Earth.

▶ **Solution:** For simplicity, let us model the Earth with a sphere of radius R_E (this simplification is sufficient for understanding the determination of Earth's mass). The acceleration of gravity, g_E, at its surface is simply

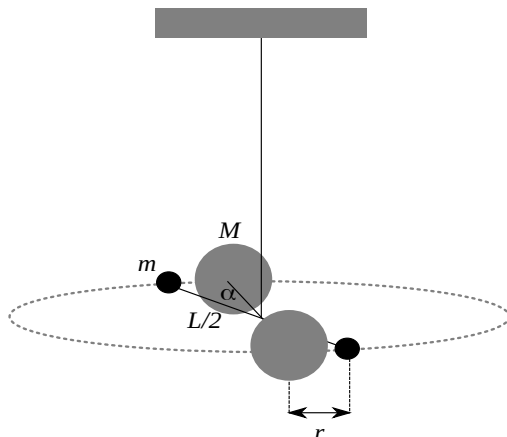$$g_E = G \frac{M_E}{R_E^2} \tag{11.17}$$

Figure 11.1: Schematic illustration of the Cavendish experiment, that determines the magnitude of gravitational constant G.

which implies that the mass of the Earth is just

$$M_E = g_E \frac{R_E^2}{G}. \tag{11.18}$$

We know G and R_E, but we have to find a way to determine g_E. Let us consider, again, a simple pendulum experiment. We have a small ball of mass m attached to massless rope of length l (see Fig.11.2).

The equation of motion of the pendulum is easily determined from the Lagrangian L, that in the case of our pendulum reads (note $I = \frac{1}{2}m\,l^2$)

$$
\begin{aligned}
L &= \frac{1}{2}v^2 + \frac{1}{2}I\omega^2 - g_E m\,h & (11.19) \\
&= \frac{3}{4}ml^2\omega^2 - g_E\,m\,l(1 - \cos\varphi). & (11.20)
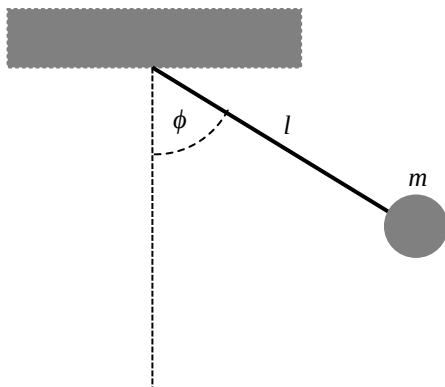\end{aligned}
$$

Figure 11.2: A schematic figure illustrating configuration of the pendulum, to determine gravitational acceleration, g_E, at Earth's surface.

The Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} = 0 \qquad (11.21)$$

together with Lagrangian (11.20) to equations of motion

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\frac{2}{3}\frac{g_E}{l}\sin\varphi. \qquad (11.22)$$

For small values of φ last equation will read

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} \simeq -\frac{2}{3}\frac{g_E}{l}\varphi. \qquad (11.23)$$

Of course, this is the equation for harmonic motion. With initial conditions $\varphi(t=0) = \varphi_0$ and $\dot{\varphi}(t=0) = 0$ we arrive to solution

$$\varphi(t) = \varphi_0 \cos\sqrt{\frac{2g_E}{3l}}t \qquad (11.24)$$

with angular frequency

$$\omega = \sqrt{\frac{2g_E}{3l}}.$$ (11.25)

Corresponding period T of pendulum motion reads

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{3l}{2g_E}}.$$ (11.26)

The gravitational acceleration at Earth's surface is determined by the formula

$$g_E = 4\pi^2\frac{l}{T^2}.$$ (11.27)

Putting last the formula into equation (11.18) we obtain relation that determines mass of the Earth

$$M_E = 4\pi^2\frac{R_E^2}{G}\frac{l}{T^2}.$$ (11.28)

E3 Determine mass of the Sun and Planets from third Keplerian law.

▶ **Solution:** The Sun and a planet revolve around mutual center of mass that determine the trajectory of the planet. It is an ellipse with major semiaxis a_P. If the period of planet motion is T then the third Keplerian law states

$$\frac{a_P^3}{T_P} = \frac{1}{4\pi^2}(M_S + M_P).$$ (11.29)

Now, we already know the mass of one of the planets, the Earth. From equation (11.29) we determine mass of the Sun, M_S, in terms of Earth orbital parameters, i.e. we get the formula

$$M_S = 4\pi^2\frac{a_E^3}{T_E^2} - M_E.$$ (11.30)

Once we know the mass of the Sun, we determine, masses of other planets. The corresponding formula reads

$$M_P = 4\pi^2\frac{a_P^3}{T_P^2} - M_S.$$ (11.31)

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